

# On the virtual levels of positively projected massless Coulomb-Dirac operators

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## Abstract

Considering different self-adjoint realisations of positively projected massless Coulomb-Dirac operators we find out, under which conditions any negative perturbation, however small, leads to emergence of negative spectrum. We also prove some weighted Lieb-Thirring estimates for negative eigenvalues of such operators. In the process we find explicit spectral representations for all self-adjoint realisations of massless Coulomb-Dirac operators on the half-line.

## 1 Introduction

In a related paper [11] the authors have obtained estimates of Cwikel-Lieb-Rozenblum and Lieb-Thirring types on the negative eigenvalues of the perturbed positively projected two-dimensional massless Coulomb-Dirac operator emerging in the study of graphene. For the critical value of the coupling constant the control on the number of eigenvalues failed, which naturally posed a question about the existence of a virtual level at zero, i.e. the situation when every non-trivial negative perturbation leads to emergence of negative spectrum.

Trying to resolve this question we arrived at the study of Coulomb-Dirac operators on the half-line  $\mathbb{R}_+$  associated to the differential expression

$$d^{\nu,\kappa} := \begin{pmatrix} -\nu/r & -\frac{d}{dr} - \frac{\kappa}{r} \\ \frac{d}{dr} - \frac{\kappa}{r} & -\nu/r \end{pmatrix}. \quad (1.1)$$

Here  $\nu \in \mathbb{R}$  is the strength of the Coulomb potential (nuclear charge) and  $\kappa \in \mathbb{R}$  is a parameter typically arising after the separation of angular motion in several dimensions (see [17]). Typically  $\kappa$  takes integer or half-integer values, but we will not need this assumption. Throughout the text we use the notation

$$\beta := \sqrt{\kappa^2 - \nu^2} \in \overline{\mathbb{R}_+} \cup i\mathbb{R}_+.$$

It turns out that for  $\beta \geq 1/2$  the operator  $D^{\nu,\kappa}$  defined by (1.1) on  $C_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$  is essentially self-adjoint in  $L^2(\mathbb{R}_+, \mathbb{C}^2)$ . Otherwise, there exist a one parameter family  $D_\theta^{\nu,\kappa}$ ,  $\theta \in [0, \pi)$ , of self-adjoint extensions of  $D^{\nu,\kappa}$  differing by boundary conditions at zero. This is the result of Theorem 1, which is essentially contained

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in [20]. We denote the set of triples  $(\nu, \kappa, \theta)$  for which  $D_\theta^{\nu, \kappa}$  is defined in Theorem 1 by  $\mathfrak{M}$ .

In Theorem 2 for every  $(\nu, \kappa, \theta) \in \mathfrak{M}$  we obtain the spectral representation of  $D_\theta^{\nu, \kappa}$ , namely we find an explicit unitary operator  $\mathcal{U}_\theta^{\nu, \kappa} : \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2, dr) \rightarrow \mathbb{L}^2(\mathbb{R}, \mathbb{C}, dx)$  such that  $\mathcal{U}_\theta^{\nu, \kappa} D_\theta^{\nu, \kappa} (\mathcal{U}_\theta^{\nu, \kappa})^*$  is the operator of multiplication by the independent variable. In particular, for all  $(\nu, \kappa, \theta) \in \mathfrak{M}$  the spectrum of  $D_\theta^{\nu, \kappa}$  is purely absolutely continuous, simple and coincides with  $\mathbb{R}$ . The existence and general form of spectral representations for one-dimensional Dirac systems is already proved in Theorem 9.7 of [21], which provides a construction of the spectral representation with respect to a matrix-valued measure given by an explicit formula. Proving that this measure is of rank one and is mutually absolutely continuous with respect to the Lebesgue measure requires considerable work. Our proof of Theorem 2 is not based on a direct application of this general result, since such an application seems to be more involved than our approach tailored specifically to  $D_\theta^{\nu, \kappa}$ . A related result for Coulomb-Dirac operators with positive mass can be found in [18].

Then we study the negative spectrum of perturbations of  $D_\theta^{\nu, \kappa}$  restricted to its positive spectral subspace. Denoting for any self-adjoint operator  $A$  and Borel subset  $\mathcal{I}$  of  $\mathbb{R}$  the corresponding spectral projector by  $P_{\mathcal{I}}(A)$  we define  $P_{\theta, \infty}^{\nu, \kappa} := P_{[0, \infty)}(D_\theta^{\nu, \kappa})$ . In  $P_{\theta, \infty}^{\nu, \kappa} \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$  we consider the operator

$$D_{\theta, \infty}^{\nu, \kappa}(V) := P_{\theta, \infty}^{\nu, \kappa}(D_\theta^{\nu, \kappa} - V)P_{\theta, \infty}^{\nu, \kappa} \quad (1.2)$$

with  $V$  being an operator of multiplication by a measurable Hermitian  $2 \times 2$ -matrix-function. For  $2 \times 2$  matrices their norms  $\|\cdot\|_{\mathbb{C}^{2 \times 2}}$ , absolute values  $|\cdot|$  and positive parts  $(\cdot)_+$  are defined in the spectral sense. We are, in particular, interested in the role which the choice of a self-adjoint realisation plays for the existence of a virtual level.

In Theorem 3 we obtain three types of results concerning virtual levels: The parameter set  $\mathfrak{M}$  is a disjoint union of  $\mathfrak{M}_I$ ,  $\mathfrak{M}_{II}$  and  $\mathfrak{M}_{III}$  such that

- I. For  $(\nu, \kappa, \theta) \in \mathfrak{M}_I$  the operator  $D_{\theta, \infty}^{\nu, \kappa}(0)$  has a virtual level at zero, i.e. there exists a measurable function  $A_\theta^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \mathbb{C}^2$  vanishing almost nowhere such that for any  $V$  satisfying

$$\int_0^\infty \langle A_\theta^{\nu, \kappa}(r), V(r) A_\theta^{\nu, \kappa}(r) \rangle_{\mathbb{C}^2} dr > 0 \quad (1.3)$$

and any  $\alpha > 0$  the operator  $D_{\theta, \infty}^{\nu, \kappa}(\alpha V)$  has non-empty negative spectrum. Condition (1.3) is satisfied for all  $V \geq 0$  which are positive definite on some sets of positive Lebesgue measure.

- II. For  $(\nu, \kappa, \theta) \in \mathfrak{M}_{II}$  and  $q > 1$  there exist weight functions  $W_{\theta, q}^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}_+}$  such that the inequality

$$\text{rank } P_{(-\infty, 0)}(D_{\theta, \infty}^{\nu, \kappa}(V)) \leq \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^q W_{\theta, q}^{\nu, \kappa}(r) dr \quad (1.4)$$

holds. Consequently, for any  $V$  for which the right hand side of (1.4) is finite, the operator  $D_{\theta, \infty}^{\nu, \kappa}(\alpha V)$  has no negative spectrum provided  $|\alpha|$  is small enough.

III. For  $(\nu, \kappa, \theta) \in \mathfrak{M}_{\text{III}}$  and  $V_+ \in L^\infty(\mathbb{R}_+, \mathbb{C}^{2 \times 2})$  the estimate

$$\begin{aligned} & \text{rank } P_{(-\infty, 0)}(D_{\theta, \infty}^{\nu, \kappa}(V)) \\ & \leq K^{\nu, \kappa} \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}} \left( \ln^2(e^{\tan \theta} r) + \ln^2(e + 2r\|V_+\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^{2 \times 2})}) \right) dr \end{aligned} \quad (1.5)$$

holds with a finite constant  $K^{\nu, \kappa}$ . Again, as soon as the right hand side of (1.5) is finite, the operator  $D_{\theta, \infty}^{\nu, \kappa}(\alpha V)$  has no negative spectrum provided  $|\alpha|$  is small enough.

The questions of existence of a virtual level at zero are already studied for different self-adjoint operators, see e.g. [15, 19].

In Theorem 4 we obtain estimates of Lieb-Thirring type (see [10] for the original result and [9, 6] for reviews of further developments):

a) For most  $(\nu, \kappa, \theta) \in \mathfrak{M}$  we can prove for any  $\gamma > 0$

$$\text{tr} (D_{\theta, \infty}^{\nu, \kappa}(V))_-^\gamma \leq K^{\nu, \kappa, \gamma} \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^{1+\gamma} W_\theta^{\nu, \kappa}(r) dr \quad (1.6)$$

with an appropriate weight function  $W_\theta^{\nu, \kappa} \geq 0$  and  $K^{\nu, \kappa, \gamma} \in \mathbb{R}_+$ . In many cases we have  $W_\theta^{\nu, \kappa} \equiv 1$ .

b) In the special case  $\beta \in (0, 1/2)$ ,  $\theta = 0$  we need to assume  $\gamma > 2\beta$  and replace (1.6) with

$$\begin{aligned} & \text{tr} (D_{0, \infty}^{\nu, \kappa}(V))_-^\gamma \\ & \leq K^{\nu, \kappa, \gamma} \left( \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^{1+\gamma-2\beta} r^{-2\beta} dr + \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^{1+\gamma} dr \right), \end{aligned} \quad (1.7)$$

where  $K^{\nu, \kappa, \gamma}$  is a finite constant.

The ranges of applicability of the above results are represented in the following table:

	$\beta \in i\mathbb{R}_+$	$\beta = 0 = \kappa$	$\beta = 0 \neq \kappa$	$\beta \in (0, 1/2)$	$\beta \geq 1/2$
$\theta = 0$	Ia1	Ia1	IIIa	Ib	—
$\theta = \pi/2$	Ia1	Ia1	Ia1	IIa1	IIa1
$\theta \in (0, \pi) \setminus \{\frac{\pi}{2}\}$	Ia1	Ia1	IIIa	IIa	—

Table 1

Here the Roman numbers indicate the applicable part of Theorem 3, the letters the part of Theorem 4 and “1” indicates that (1.6) holds with  $W_\theta^{\nu, \kappa} \equiv 1$ . In the cases marked with “—” there exists no self-adjoint realisation, see Theorem 1. Note that in the case “Ia1” inequality (1.6) is a form of the Hardy-Lieb-Thirring inequality (see [3, 4, 7]).

At last, in Theorem 5 we apply our results to the two-dimensional massless Coulomb-Dirac operator answering the question stated at the beginning of the introduction. A further application to three-dimensional massless Coulomb-Dirac operators can be obtained analogously.

In the following section we explicitly formulate our main results. Their proofs constitute the rest of the article.

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## 2 Main results

### 2.1 Self-adjoint realisations of Coulomb-Dirac operators on the half-line

First we introduce a family of self-adjoint operators in  $L^2(\mathbb{R}_+, \mathbb{C}^2)$  corresponding to the differential expression (1.1). We start from the symmetric operator  $D^{\nu, \kappa}$  defined on  $C_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$ . Then the action of the adjoint operator  $(D^{\nu, \kappa})^*$  is also given by (1.1), but on the “maximal” domain

$$\mathfrak{D}((D^{\nu, \kappa})^*) = \{f \in L^2(\mathbb{R}_+, \mathbb{C}^2) : f \in \text{AC}_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^2), d^{\nu, \kappa} f \in L^2(\mathbb{R}_+, \mathbb{C}^2)\}.$$

In the following theorem we characterise all self-adjoint extensions of  $D^{\nu, \kappa}$ . We will make use of the functions  $\Psi_M^{\nu, \kappa}$  and  $\Psi_U^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \mathbb{C}^2$  introduced in (3.1) and (3.2).

**Theorem 1.**

1. For  $\beta \in [0, 1/2)$  and  $\kappa \neq 0$  every self-adjoint extension of  $D^{\nu, \kappa}$  coincides with  $D_\theta^{\nu, \kappa}$  with some  $\theta \in [0, \pi)$ , where  $D_\theta^{\nu, \kappa}$  is the restriction of  $(D^{\nu, \kappa})^*$  to the set of functions  $f \in \mathfrak{D}((D^{\nu, \kappa})^*)$  satisfying the boundary condition

$$\lim_{r \rightarrow +0} \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f(r), \cos \theta \Psi_U^{\nu, \kappa}(r) + \sin \theta \Psi_M^{\nu, \kappa}(r) \right\rangle_{\mathbb{C}^2} = 0. \quad (2.1)$$

2. For  $\beta \geq 1/2$  the operator  $D^{\nu, \kappa}$  is essentially self-adjoint. We denote its closure by  $D_{\pi/2}^{\nu, \kappa}$ . Every  $f \in \mathfrak{D}(D_{\pi/2}^{\nu, \kappa})$  satisfies (2.1) with  $\theta := \pi/2$ .
3. For  $\beta \in i\mathbb{R}_+$  or  $\kappa = 0$  every self-adjoint extension of  $D^{\nu, \kappa}$  coincides with  $D_\theta^{\nu, \kappa}$  with some  $\theta \in [0, \pi)$ , where  $D_\theta^{\nu, \kappa}$  is the restriction of  $(D^{\nu, \kappa})^*$  to the set of functions  $f \in \mathfrak{D}((D^{\nu, \kappa})^*)$  satisfying the boundary condition

$$\lim_{r \rightarrow +0} \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f(r), e^{i\theta} \Psi_U^{\nu, \kappa}(r) + e^{-i\theta} \Psi_M^{\nu, \kappa}(r) \right\rangle_{\mathbb{C}^2} = 0. \quad (2.2)$$

### 2.2 Spectral representation of $D_\theta^{\nu, \kappa}$

From now on we study the self-adjoint operator  $D_\theta^{\nu, \kappa}$  defined in Theorem 1. Using the auxiliary functions and constants introduced in Lemmata 9, 10, 12 and 16 we are able to find an explicit transform which delivers the spectral representation for  $D_\theta^{\nu, \kappa}$ :

**Theorem 2.** Let  $\Lambda$  be the operator of multiplication by the independent variable in  $L^2(\mathbb{R}, \mathbb{C}, dx)$ . The unitary operator  $\mathcal{U}_\theta^{\nu, \kappa} : L^2(\mathbb{R}_+, \mathbb{C}^2, dr) \rightarrow L^2(\mathbb{R}, \mathbb{C}, dx)$  given by

$$\mathcal{U}_\theta^{\nu, \kappa} f := \lim_{R \rightarrow \infty}^2 \sqrt{m_\theta^{\nu, \kappa}(\cdot)} \int_{1/R}^R (\Phi_{0, \theta}^{\nu, \kappa}(\cdot; y))^\top f(y) dy \quad (2.3)$$

delivers the spectral representation of  $D_\theta^{\nu, \kappa}$ , i.e.

$$D_\theta^{\nu, \kappa} = (\mathcal{U}_\theta^{\nu, \kappa})^* \Lambda \mathcal{U}_\theta^{\nu, \kappa} \quad (2.4)$$

holds.

### 2.3 Existence of virtual levels for positively projected massless Coulomb-Dirac operators on the half-line

Let  $V$  be a measurable Hermitian  $2 \times 2$ -matrix-function on  $\mathbb{R}_+$ . We assume

**Hypothesis A.** *The operator  $P_{\theta,\infty}^{\nu,\kappa} V P_{\theta,\infty}^{\nu,\kappa}$  is relatively form bounded with respect to  $D_{\theta}^{\nu,\kappa}$  with a form bound less than one in  $P_{\theta,\infty}^{\nu,\kappa} \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ .*

Under Hypothesis A the operator (1.2) is well defined in the form sense and self-adjoint by the KLMN theorem (see e.g. Theorem X.17 in [14]). Hypothesis A is trivially satisfied for  $V \in \mathbb{L}^\infty(\mathbb{R}_+)$ .

The main result concerning the existence of a virtual level is the following theorem.

**Theorem 3.**

I. Let  $A_{\theta}^{\nu,\kappa} : \mathbb{R}_+ \rightarrow \mathbb{C}^2$  be defined by

$$A_{\theta}^{\nu,\kappa}(r) := \begin{cases} e^{i\theta} r^{i\nu} \begin{pmatrix} 1 \\ i \end{pmatrix} + e^{-i\theta} r^{-i\nu} \begin{pmatrix} 1 \\ -i \end{pmatrix}, & \text{for } \kappa = 0, \nu \in \mathbb{R}, \theta \in [0, \pi); \\ \begin{pmatrix} -\nu \\ \kappa \end{pmatrix}, & \text{for } \kappa \neq 0, \beta = 0, \theta = \pi/2; \\ e^{i\theta} r^{-\beta} \kappa \begin{pmatrix} \kappa - \beta \\ -\nu \end{pmatrix} + e^{-i\theta} r^{\beta} \kappa \begin{pmatrix} \kappa + \beta \\ -\nu \end{pmatrix}, & \text{for } \kappa \neq 0, \beta \in i\mathbb{R}_+, \theta \in [0, \pi); \\ r^{-\beta} \begin{pmatrix} \nu \\ -\kappa - \beta \end{pmatrix}, & \text{for } \beta \in (0, 1/2), \theta = 0. \end{cases} \quad (2.5)$$

Let  $\mathfrak{M}_I$  denote the set of all triples  $(\nu, \kappa, \theta)$  from the right hand side of (2.5). For any  $(\nu, \kappa, \theta) \in \mathfrak{M}_I$  assume that  $V$  satisfies

$$\|V\|_{\mathbb{C}^{2 \times 2}} \in \mathbb{L}^1(\mathbb{R}_+, r^{2-2\operatorname{Re} \beta} dr) \text{ and } \int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), |V(r)| A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr < \infty. \quad (2.6)$$

Then the negative spectrum of  $D_{\theta,\infty}^{\nu,\kappa}(V)$  is non-empty provided (1.3) holds.

II. Let  $\mathfrak{M}_{II}$  be the set of  $(\nu, \kappa, \theta) \in \mathfrak{M}$  such that either  $\beta > 0$  and  $\theta = \pi/2$ , or  $\beta \in (0, 1/2)$  and  $\theta \in (0, \pi) \setminus \{\pi/2\}$  holds. For all  $(\nu, \kappa, \theta) \in \mathfrak{M}_{II}$  and any  $q \in (1, 1 + 2\beta)$  there exists  $C_q^{\nu,\kappa} > 0$  such that (1.4) holds with

$$W_{\theta,q}^{\nu,\kappa}(r) := C_q^{\nu,\kappa} \begin{cases} |\cot \theta|^{1+(q-1)/(2\beta)} r^{-2\beta}, & \text{for } r \leq |\cot \theta|^{1/(2\beta)}; \\ r^{q-1}, & \text{for } r \geq |\cot \theta|^{1/(2\beta)}. \end{cases} \quad (2.7)$$

Finiteness of the right hand side of (1.4) implies Hypothesis A for  $V := V_+$ .

III. Let  $V_+ \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{C}^{2 \times 2})$ . Let  $\mathfrak{M}_{III}$  be the set of  $(\nu, \kappa, \theta)$  such that  $\kappa^2 = \nu^2 \neq 0$  and  $\theta \in [0, \pi) \setminus \{\pi/2\}$  holds. For all  $(\nu, \kappa, \theta) \in \mathfrak{M}_{III}$  there exists a finite constant  $K^{\nu,\kappa}$  independent of  $\theta$  such that the estimate (1.5) holds.

### 2.4 Lieb-Thirring type estimates on the negative eigenvalues of $D_{\theta,\infty}^{\nu,\kappa}(V)$

In Theorem 4 we provide estimates on the sums of powers of negative eigenvalues of  $D_{\theta,\infty}^{\nu,\kappa}(V)$  via weighted integrals of powers of the perturbation potential  $V$ . In the case of  $\theta = \pi/2$  the result is fully analogous to the classical Lieb-Thirring estimate.

**Theorem 4.**

a) For  $\nu, \kappa \in \mathbb{R}$ ,  $\theta \in [0, \pi)$  let  $W_{\theta}^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be defined by

$$W_{\theta}^{\nu, \kappa}(r) := \begin{cases} 1, & \begin{cases} \text{for } \nu^2 = \kappa^2 \neq 0 \text{ and } \theta = \pi/2; \\ \text{for } \beta \in i\mathbb{R}_+ \text{ and } \theta \in [0, \pi); \\ \text{for } \kappa = 0, \nu \in \mathbb{R} \text{ and } \theta \in [0, \pi); \end{cases} \\ \max \{ -\ln(e^{\tan \theta} r), 1 \}^2, & \text{for } \nu^2 = \kappa^2 \neq 0 \text{ and } \theta \in [0, \pi) \setminus \{\pi/2\}; \\ \max \{ 1, |\cot \theta| r^{-2\beta} \}, & \begin{cases} \text{for } \beta \in (0, 1/2) \text{ and } \theta \in (0, \pi); \\ \text{for } \beta \geq 1/2 \text{ and } \theta = \pi/2. \end{cases} \end{cases} \quad (2.8)$$

Then for any  $\gamma > 0$  there exists  $K^{\nu, \kappa, \gamma} > 0$  such that (1.6) holds.

b) For  $\beta \in (0, 1/2)$  and any  $\gamma > 2\beta$  there exists  $K^{\nu, \kappa, \gamma} > 0$  such that (1.7) holds.

In both cases the finiteness of the right hand sides of (1.6) or (1.7) implies that  $V := V_+$  satisfies Hypothesis A.

## 2.5 Application to two-dimensional projected Coulomb-Dirac operators

Let us now consider positively projected massless Coulomb-Dirac operators in two dimensions. Such operators are relevant for description of graphene with Coulomb impurity [11]. Every  $u \in \mathbf{L}^2(\mathbb{R}^2)$  can be represented in the polar coordinates as

$$u(r, \varphi) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} r^{-1/2} u_m(r) e^{im\varphi}$$

with

$$u_m(r) := \sqrt{\frac{r}{2\pi}} \int_0^{2\pi} u(r, \varphi) e^{-im\varphi} d\varphi.$$

Introducing the unitary angular momentum decomposition

$$\mathcal{A} : \mathbf{L}^2(\mathbb{R}^2, \mathbb{C}^2) \rightarrow \bigoplus_{\varkappa \in \mathbb{Z} + 1/2} \mathbf{L}^2(\mathbb{R}_+, \mathbb{C}^2), \quad \begin{pmatrix} v \\ w \end{pmatrix} \mapsto \bigoplus_{\varkappa \in \mathbb{Z} + 1/2} \begin{pmatrix} v_{\varkappa-1/2} \\ iw_{\varkappa+1/2} \end{pmatrix} \quad (2.9)$$

we observe (see e.g. [11]) that for any self-adjoint realisation of  $-i\boldsymbol{\sigma} \cdot \nabla - \nu|\cdot|^{-1}$  in  $\mathbf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$  there exists a map

$$\boldsymbol{\theta} : \mathbb{Z} + 1/2 \rightarrow [0, \pi) \quad \text{with} \quad \boldsymbol{\theta}(\kappa) = \pi/2 \quad \text{for all} \quad \kappa^2 \geq \nu^2 + 1/4 \quad (2.10)$$

such that the self-adjoint operator in question coincides with

$$D_{\boldsymbol{\theta}}^{\nu} := \mathcal{A}^* \left( \bigoplus_{\kappa \in \mathbb{Z} + 1/2} D_{\boldsymbol{\theta}(\kappa)}^{\nu, \kappa} \right) \mathcal{A}, \quad (2.11)$$

where the components on the right hand side are defined in Theorem 1. On the other hand, every  $\boldsymbol{\theta}$  satisfying (2.10) gives rise to a self-adjoint realisation of  $-i\boldsymbol{\sigma} \cdot \nabla - \nu|\cdot|^{-1}$  in  $\mathbf{L}^2(\mathbb{R}^2, \mathbb{C}^2)$  via (2.11).

Let  $P_\theta^\nu := \mathcal{A}^*(\bigoplus_{\kappa \in \mathbb{Z}+1/2} P_{\theta,\infty}^{\nu,\kappa})\mathcal{A}$  be the spectral projector onto the positive spectral subspace of  $D_\theta^\nu$ . For measurable Hermitian  $(2 \times 2)$ -matrix functions  $Q$  on  $\mathbb{R}^2$  we are interested in the negative spectrum of

$$D_\theta^\nu(Q) := P_\theta^\nu(D_\theta^\nu - Q)P_\theta^\nu \quad (2.12)$$

in  $P_\theta^\nu \mathbb{L}^2(\mathbb{R}^2, \mathbb{C}^2)$ . We assume

**Hypothesis B.** *The operator  $P_\theta^\nu Q P_\theta^\nu$  is relatively form bounded with respect to  $D_\theta^\nu$  with a relative bound less than one in  $P_\theta^\nu \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ .*

Under Hypothesis B the operator (2.12) is well-defined via its quadratic form and self-adjoint by KLMN theorem. Hypothesis B is fulfilled for all  $Q \in \mathbb{L}^\infty(\mathbb{R}^2)$ .

**Theorem 5.** *Let  $\nu \in \mathbb{R}$ .*

1. *Suppose that there exists  $\kappa_0 \in \mathbb{Z} + 1/2$  such that  $(\nu, \kappa_0, \theta(\kappa_0)) \in \mathfrak{M}_I$  as defined in Theorem 3. Then  $D_\theta^\nu(Q)$  has non-empty negative spectrum provided*

$$V := \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} Q_{11}(\cdot, \varphi) & -iQ_{12}(\cdot, \varphi)e^{i\varphi} \\ iQ_{21}(\cdot, \varphi)e^{-i\varphi} & Q_{22}(\cdot, \varphi) \end{pmatrix} d\varphi \quad (2.13)$$

*satisfies (2.6) and (1.3) with  $(\nu, \kappa, \theta) := (\nu, \kappa_0, \theta(\kappa_0))$ .*

2. *Suppose that for all  $\kappa \in \mathbb{Z} + 1/2$  the triple  $(\nu, \kappa, \theta(\kappa))$  does not belong to  $\mathfrak{M}_I$ . If there exists a measurable function  $R : \mathbb{R}_+ \rightarrow \overline{\mathbb{R}_+}$  with*

$$Q \leq R(|\cdot|)\mathbb{I} \quad \text{almost everywhere in } \mathbb{R}^2 \quad (2.14)$$

*(where  $\mathbb{I}$  denotes the  $2 \times 2$  identity matrix) such that:*

- (a) *For all  $\kappa \in \mathbb{Z} + 1/2$  with  $\kappa^2 \leq \nu^2 + 1/4$  such that  $(\nu, \kappa, \theta(\kappa)) \in \mathfrak{M}_{II}$  there exists  $q \in (1, 1 + 2\beta)$  such that the right hand side of (1.4) with  $V := R\mathbb{I}$  and  $W_{\theta,q}^{\nu,\kappa}$  defined in (2.7) is finite;*
- (b) *For all  $\kappa \in \mathbb{Z} + 1/2$  with  $\kappa^2 \leq \nu^2 + 1/4$  such that  $(\nu, \kappa, \theta(\kappa)) \in \mathfrak{M}_{III}$  we have  $R \in \mathbb{L}^\infty(\mathbb{R}_+)$  and the right hand side of (1.5) with  $V := R\mathbb{I}$  is finite;*
- (c)  *$R \in \mathbb{L}^\infty(\mathbb{R}_+, r dr) + \mathbb{L}^2(\mathbb{R}_+, r dr)$*

*then there exists  $\alpha_c > 0$  such that for all  $\alpha \in [0, \alpha_c)$  operator  $D_\theta^\nu(\alpha Q)$  has no negative spectrum. Note that for any  $Q \in \mathbb{C}_0^\infty(\mathbb{R}^2)$  all the assumptions are satisfied.*

For  $\nu \in [0, 1/2]$  there exists a distinguished self-adjoint realisation  $D^\nu$  of  $-i\sigma \cdot \nabla - \nu|\cdot|^{-1}$ , which coincides with  $D_{\theta_0}^\nu$  with  $\theta_0(\kappa) := \pi/2$  for all  $\kappa \in \mathbb{Z} + 1/2$ , see [12, 11]. We write  $D^\nu(Q)$  for  $D_{\theta_0}^\nu(Q)$ . Theorem 5 implies

**Corollary 6.** *Suppose that  $V$  defined in (2.13) satisfies*

$$\|V\|_{\mathbb{C}^{2 \times 2}} \in \mathbb{L}^1(\mathbb{R}_+, (1+r^2)dr) \quad \text{and} \quad \int_0^\infty \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, V(r) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} dr > 0.$$

*Then for any  $\alpha > 0$  the negative spectrum of  $D^{1/2}(\alpha Q)$  is non-empty.*

For  $\nu \in [0, 1/2)$  an application of Theorem 5, part 2 to  $D^\nu(Q)$  gives a weaker result than that of Theorem 3 in [11], which gives the following bound on the amount of negative eigenvalues of  $D^\nu(Q)$ : There exists  $C_\nu^{\text{CLR}} > 0$  such that

$$\text{rank}(D^\nu(Q))_- \leq C_\nu^{\text{CLR}} \int_{\mathbb{R}^2} \text{tr}(Q_+(\mathbf{x}))^2 d\mathbf{x}. \quad (2.15)$$

Indeed, even for  $Q := R(|\cdot|)\mathbb{I}$  with radial non-negative  $R$  the hypothesis of Theorem 5 requires  $R \in \mathbf{L}^\infty(\mathbb{R}_+, r dr) + \mathbf{L}^2(\mathbb{R}_+, r dr)$  and

$$\int_0^\infty R^q(r) r^{q-1} dr < \infty \quad \text{for some } q \in (1, 1 + 2\sqrt{1/4 - \nu^2}).$$

But then we can write  $R = R_1 + R_2$  with  $R_1 \in \mathbf{L}^\infty(\mathbb{R}_+, r dr)$ ,  $R_2 \in \mathbf{L}^2(\mathbb{R}_+, r dr)$  and  $R_1, R_2 \geq 0$ . Hence we have

$$\begin{aligned} \int_{\mathbb{R}^2} \text{tr}(Q_+(\mathbf{x}))^2 d\mathbf{x} &= 4 \int_0^\infty (R_1(r) + R_2(r))^2 r dr \\ &\leq 8 \|R_1\|_{\mathbf{L}^\infty(\mathbb{R}_+, r dr)}^{2-q} \int_0^\infty R_1^q(r) r^{q-1} dr + 8 \int_0^\infty R_2^2(r) r dr < \infty, \end{aligned}$$

so (2.15) implies the statement of Theorem 5, part 2 under weaker assumptions.

### 3 Proof of Theorem 1

The fundamental solution of  $d^{\nu, \kappa} \Psi^{\nu, \kappa} = 0$  is a linear combination of  $\Psi_M^{\nu, \kappa}$  and  $\Psi_U^{\nu, \kappa}$  with

$$\Psi_M^{\nu, \kappa}(r) := \begin{cases} \kappa r^\beta \begin{pmatrix} \kappa + \beta \\ -\nu \end{pmatrix}, & \text{for } \nu^2 \neq \kappa^2 \neq 0 \text{ and } \kappa \neq -\beta; \\ \beta r^\beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \text{for } \beta = -\kappa \in \mathbb{R}_+; \\ \begin{pmatrix} -\nu \\ \kappa \end{pmatrix}, & \text{for } \nu^2 = \kappa^2 \neq 0; \\ r^{-i\nu} \begin{pmatrix} 1 \\ -i \end{pmatrix}, & \text{for } \kappa = 0 \end{cases} \quad (3.1)$$

and

$$\Psi_U^{\nu, \kappa}(r) := \begin{cases} r^{-\beta} \begin{pmatrix} \nu \\ -\kappa - \beta \end{pmatrix}, & \text{for } \beta \in \mathbb{R}_+ \text{ and } \kappa \neq -\beta; \\ r^{-\beta} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{for } \beta = -\kappa \in \mathbb{R}_+; \\ \kappa r^{-\beta} \begin{pmatrix} \kappa - \beta \\ -\nu \end{pmatrix}, & \text{for } \beta \in i\mathbb{R}_+ \text{ and } \kappa \neq 0; \\ \ln r \begin{pmatrix} -\nu \\ \kappa \end{pmatrix} - \frac{1}{2\kappa} \begin{pmatrix} \nu \\ \kappa \end{pmatrix}, & \text{for } \nu^2 = \kappa^2 \neq 0; \\ r^{i\nu} \begin{pmatrix} 1 \\ i \end{pmatrix}, & \text{for } \kappa = 0. \end{cases} \quad (3.2)$$

Since none of the solutions belongs to  $\mathbf{L}^2((1, \infty))$ , (1.1) is always in the limit point case at infinity. For  $\beta \geq 1/2$ ,  $\Psi_U^{\nu, \kappa}$  does not belong to  $\mathbf{L}^2((0, 1))$  and we have a limit point case at zero. Otherwise there is a limit circle case at zero. The rest follows from Theorem 1.5 in [20].

### 4 Proof of Theorem 2

We begin by studying the classical solutions of the spectral equation

$$d^{\nu, \kappa} f = \lambda f \quad (4.1)$$

on the half-line  $\mathbb{R}_+$ , where  $\lambda \in \mathbb{C} \setminus i\overline{\mathbb{R}_+}$  is the spectral parameter.



**Lemma 7.** *Let  $M$  and  $U$  be the Kummer functions (see [1], Section 13.2). For  $\lambda = 1$  any classical solution of (4.1) is a linear combination of the following two linearly independent functions on  $\mathbb{R}_+$ :*

1. For  $\nu^2 \neq \kappa^2 \neq 0$  and  $\beta \neq -\kappa$ ,

$$\begin{aligned} \Phi_M^{\nu, \kappa}(r) := & r^\beta e^{-ir} \left( \beta(\kappa + \beta + i\nu) M(i\nu + \beta, 2\beta, 2ir) \begin{pmatrix} 1 \\ -i \end{pmatrix} \right. \\ & \left. + (\nu - i\beta) M(1 + i\nu + \beta, 1 + 2\beta, 2ir) \begin{pmatrix} \nu \\ -\kappa - \beta \end{pmatrix} \right) \end{aligned}$$

and

$$\begin{aligned} \tilde{\Phi}_U^{\nu, \kappa}(r) := & r^\beta e^{-ir} \left( (\kappa + \beta + i\nu) U(i\nu + \beta, 2\beta, 2ir) \begin{pmatrix} 1 \\ -i \end{pmatrix} \right. \\ & \left. + 2(i\beta - \nu) U(1 + i\nu + \beta, 1 + 2\beta, 2ir) \begin{pmatrix} \nu \\ -\kappa - \beta \end{pmatrix} \right); \end{aligned} \quad (4.2)$$

2. for  $\beta = -\kappa \in \mathbb{R}_+$ ,

$$\Phi_M^{\nu, \kappa}(r) := \beta r^\beta e^{-ir} \left( M(\beta, 2\beta, 2ir) \begin{pmatrix} i \\ 1 \end{pmatrix} + M(1 + \beta, 1 + 2\beta, 2ir) \begin{pmatrix} -i \\ 0 \end{pmatrix} \right),$$

and

$$\tilde{\Phi}_U^{\nu, \kappa}(r) := r^\beta e^{-ir} \left( U(\beta, 2\beta, 2ir) \begin{pmatrix} i \\ 1 \end{pmatrix} + U(1 + \beta, 1 + 2\beta, 2ir) \begin{pmatrix} 2i\beta \\ 0 \end{pmatrix} \right) \quad (4.3)$$

3. for  $\nu^2 = \kappa^2 \neq 0$ ,

$$\begin{aligned} \Phi_M^{\nu, \kappa}(r) := & e^{-ir} \left( M(1 + i\nu, 2, 2ir) \left( (\kappa + i\nu) r \begin{pmatrix} 1 \\ -i \end{pmatrix} - \begin{pmatrix} \nu \\ -\kappa \end{pmatrix} \right) \right. \\ & \left. + (\nu - i) r M(2 + i\nu, 3, 2ir) \begin{pmatrix} \nu \\ -\kappa \end{pmatrix} \right) \end{aligned}$$

and

$$\begin{aligned} \tilde{\Phi}_U^{\nu, \kappa}(r) := & e^{-ir} \left( U(1 + i\nu, 2, 2ir) \left( (\kappa + i\nu) r \begin{pmatrix} 1 \\ -i \end{pmatrix} - \begin{pmatrix} \nu \\ -\kappa \end{pmatrix} \right) \right. \\ & \left. + 2(i - \nu) r U(2 + i\nu, 3, 2ir) \begin{pmatrix} \nu \\ -\kappa \end{pmatrix} \right); \end{aligned} \quad (4.4)$$

4. for  $\kappa = 0$ ,

$$\Phi_M^{\nu, 0}(r) := r^{-i\nu} e^{-ir} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \Phi_U^{\nu, 0}(r) := r^{i\nu} e^{ir} \begin{pmatrix} 1 \\ i \end{pmatrix}. \quad (4.5)$$

It is convenient to replace  $\tilde{\Phi}_U^{\nu,\kappa}$  by another solution:  
For  $0 < \beta < 1/2$ ,

$$\Phi_U^{\nu,\kappa}(r) := \frac{-2^{2\beta-1}ie^{i\pi\beta}\Gamma(\beta+i\nu)}{\Gamma(2\beta)}\left(\tilde{\Phi}_U^{\nu,\kappa}(r) + \frac{2\Gamma(-2\beta)}{\Gamma(1-\beta+i\nu)}\Phi_M^{\nu,\kappa}(r)\right); \quad (4.6)$$

for  $\beta \geq 1/2$ ,

$$\Phi_U^{\nu,\kappa}(r) := \frac{-2^{2\beta-1}ie^{i\pi\beta}\Gamma(\beta+i\nu)}{\Gamma(2\beta)}\tilde{\Phi}_U^{\nu,\kappa}(r);$$

for  $\beta \in i\mathbb{R}_+$  and  $\kappa \neq 0$ ,

$$\Phi_U^{\nu,\kappa}(r) := \frac{-2^{2\beta-1}i\kappa(\kappa-\beta)e^{i\pi\beta}\Gamma(\beta+i\nu)}{\nu\Gamma(2\beta)}\left(\tilde{\Phi}_U^{\nu,\kappa}(r) + \frac{2\Gamma(-2\beta)}{\Gamma(1-\beta+i\nu)}\Phi_M^{\nu,\kappa}(r)\right); \quad (4.7)$$

and for  $\nu^2 = \kappa^2 \neq 0$ ,

$$\Phi_U^{\nu,\kappa}(r) := \Gamma(i\nu)\tilde{\Phi}_U^{\nu,\kappa}(r) - \left(\ln 2 + 2\gamma + \psi(1+i\nu) + \frac{i\pi}{2} + \frac{i}{2\nu}\right)\Phi_M^{\nu,\kappa}(r), \quad (4.8)$$

where  $\psi$  is the digamma function and  $\gamma$  is the Euler-Mascheroni constant. The above solutions allow unique analytic continuations from  $\mathbb{R}_+$  to  $\mathbb{C} \setminus i\overline{\mathbb{R}_+}$ . For arbitrary  $\lambda \in \mathbb{C} \setminus i\overline{\mathbb{R}_+}$  any classical solution to (4.1) is given by a linear combination of  $\Phi_M^{\nu,\kappa}(\lambda \cdot)$  and  $\Phi_U^{\nu,\kappa}(\lambda \cdot)$ .

*Proof.* The validity of Lemma 7 can be checked by a straightforward calculation using functional relations between Kummer functions and their derivatives, see e.g. [1], 13.3.13–15, 13.3.22. It is, however, more instructive to provide a derivation of the solutions, which we will now do for  $\kappa \neq 0$ .

We start by seeking the solutions to (4.1) with  $\lambda = 1$  in the form

$$f(r) = r^\beta F(r) \quad (4.9)$$

obtaining

$$\begin{pmatrix} -\nu/r - 1 & -\frac{d}{dr} - \frac{\beta+\kappa}{r} \\ \frac{d}{dr} + \frac{\beta-\kappa}{r} & -\nu/r - 1 \end{pmatrix} \begin{pmatrix} F_1(r) \\ F_2(r) \end{pmatrix} = 0. \quad (4.10)$$

Introducing

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix} := \begin{cases} \begin{pmatrix} \frac{\kappa+\beta}{2}F_1 + \frac{\nu}{2}F_2 \\ \frac{\nu}{2}F_1 - \frac{\kappa+\beta}{2}F_2 \end{pmatrix}, & \text{for } \beta \neq \kappa \neq 0, \\ \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}, & \text{for } \beta = -\kappa \neq 0 \end{cases} \quad (4.11)$$

we obtain that (4.10) is equivalent to

$$G_2 = -G_1', \quad (4.12)$$

$$G_1''(r) + \frac{2\beta}{r}G_1'(r) + \left(1 + \frac{2\nu}{r}\right)G_1(r) = 0. \quad (4.13)$$

We can now find two linearly independent solutions of (4.13) analytic in  $\mathbb{C} \setminus i\overline{\mathbb{R}_+}$ .

For  $\beta \neq 0$  making the substitution

$$G_1(r) = e^{-ir} w(2ir), \quad z := 2ir \quad (4.14)$$

we obtain

$$zw''(z) + (2\beta - z)w'(z) - (i\nu + \beta)w(z) = 0, \quad z \in \mathbb{C} \setminus \overline{\mathbb{R}_-},$$

which is the Kummer equation with parameters  $a := i\nu + \beta$ ,  $b := 2\beta$  (see e.g. [1], 13.2.1).

For  $\beta = 0$ , the substitution

$$G_1(r) = re^{-ir} v(2ir), \quad z := 2ir \quad (4.15)$$

delivers

$$zv''(z) + (2 - z)v'(z) - (1 + i\nu)v(z) = 0, \quad z \in \mathbb{C} \setminus \overline{\mathbb{R}_-},$$

i.e. the Kummer equation with parameters  $a := 1 + i\nu$ ,  $b := 2$ .

In both cases the linearly independent solutions of the Kummer equation can be chosen as the Kummer functions  $M(a, b, z)$  and  $U(a, b, z)$  (see [1], Section 13.2). Substituting back into (4.14) (or (4.15)), (4.12) (together with 13.3.15 and 13.3.22 in [1]), (4.11) and (4.9), we obtain the two independent solutions of (4.1) with  $\lambda = 1$  analytic in  $\mathbb{C} \setminus i\overline{\mathbb{R}_+}$  stated in the lemma.  $\square$

**Lemma 8.** *The solutions  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$  have the following asymptotics at zero:*

$$\Phi_M^{\nu, \kappa}(r) = \begin{cases} \kappa r^\beta \binom{\kappa + \beta}{-\nu} + O(r^{1+\beta}), & \text{for } \nu^2 \neq \kappa^2 \neq 0 \text{ and } \kappa \neq -\beta; \\ \beta r^\beta \binom{0}{1} + O(r^{1+\beta}), & \text{for } \beta = -\kappa \in \mathbb{R}_+; \\ \binom{-\nu}{\kappa} + O(r), & \text{for } \nu^2 = \kappa^2 \neq 0; \\ r^{-i\nu} \binom{1}{-i} + O(r), & \text{for } \kappa = 0; \end{cases} \quad (4.16)$$

$$\Phi_U^{\nu, \kappa}(r) = \begin{cases} r^{-\beta} \binom{\nu}{-\kappa - \beta} + O(r^{1-\beta}), & \text{for } \beta \in \mathbb{R}_+ \setminus \{1/2, -\kappa\}; \\ r^{-1/2} \binom{\nu}{-\kappa - 1/2} + O(r^{1/2} \ln r), & \text{for } \beta = 1/2 \neq -\kappa; \\ r^{-\beta} \binom{1}{0} + O(r^{1-\beta}), & \text{for } \beta = -\kappa \in \mathbb{R}_+ \setminus \{1/2\}; \\ r^{-1/2} \binom{1}{0} + O(r^{1/2} \ln r), & \text{for } \beta = -\kappa = 1/2; \\ \kappa r^{-\beta} \binom{\kappa - \beta}{-\nu} + O(r), & \text{for } \beta \in i\mathbb{R}_+ \text{ and } \kappa \neq 0; \\ \ln r \binom{-\nu}{\kappa} - \frac{1}{2\kappa} \binom{\nu}{\kappa} + O(r \ln r), & \text{for } \nu^2 = \kappa^2 \neq 0; \\ r^{i\nu} \binom{1}{i} + O(r), & \text{for } \kappa = 0. \end{cases} \quad (4.17)$$

For  $\beta \in [0, 1/2)$  and  $\kappa \neq 0$  both  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$  are real-valued, for  $\beta \in i\mathbb{R}_+$  or  $\kappa = 0$  they are complex conjugate of each other.

*Proof.* The asymptotics of  $\Phi_M^{\nu, \kappa}$  follows from the definitions of Lemma 7 and 13.2.2 of [1].

For  $\beta \in \mathbb{R}_+ \setminus (\mathbb{N}/2)$  or  $\beta \in i\mathbb{R}_+$  and  $\kappa \neq 0$ , the expansion of  $\tilde{\Phi}_U^{\nu, \kappa}$  follows from (4.2) or (4.3) and 13.2.42 in [1]. For  $\beta \in (0, 1/2)$  and  $\beta \in i\mathbb{R}_+$ , the definitions (4.6) and (4.7) are constructed in such a way that the coefficients at  $r^\beta$  in the asymptotics (4.17) are zero.

For  $\beta \in \mathbb{N}/2$  or  $\beta = 0$  and  $\kappa \neq 0$  the expansion of  $\tilde{\Phi}_U^{\nu, \kappa}$  follows from (4.2) or (4.4) and 13.2.9 in [1]. In (4.8) the linear combination of  $\tilde{\Phi}_U^{\nu, \kappa}$  and  $\Phi_M^{\nu, \kappa}$  is chosen in such a way that (4.17) holds true.

Since all the entries of (1.1) are invariant under complex conjugation, for  $\lambda \in \mathbb{R}$  the real and imaginary parts of any solution to (4.1) are again solutions to (4.1).

For  $\beta \in [0, 1/2)$  and  $\kappa \neq 0$ , the imaginary parts of  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$  are solutions of (4.1) with  $\lambda = 1$ , thus linear combinations of  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$ . On the other hand, by (4.16) and (4.17) this imaginary parts are  $O(r^{1-\beta})$ . Hence they must vanish identically. Analogously, for  $\beta \in i\mathbb{R}_+$  or  $\kappa = 0$ , the imaginary part of  $\Phi_M^{\nu, \kappa} + \Phi_U^{\nu, \kappa}$  and the real part of  $\Phi_M^{\nu, \kappa} - \Phi_U^{\nu, \kappa}$  are solutions of (4.1) with  $\lambda = 1$ , thus linear combinations of  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$ . Since this combinations are  $O(r)$  by (4.16) and (4.17), they must be identically zero. Thus  $\Phi_U^{\nu, \kappa} = \overline{\Phi_M^{\nu, \kappa}}$  holds.  $\square$

**Lemma 9.** For  $\kappa \neq 0$ ,  $\nu \in \mathbb{R}$  and  $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+)$  let

$$c^{\nu, \kappa}(\lambda) := \begin{cases} c_{+, \pm}^{\nu, \kappa}, & \text{for } \pm \operatorname{Re} \lambda > 0, \operatorname{Im} \lambda > 0; \\ c_-^{\nu, \kappa}, & \text{for } \operatorname{Im} \lambda < 0 \end{cases} \quad (4.18)$$

with

$$c_{+, \pm}^{\nu, \kappa} := \begin{cases} \frac{i2^{2\beta-1} |\Gamma(\beta + i\nu)|^2 e^{\pm \pi \nu} e^{(1 \mp 1)i\pi\beta}}{\beta \Gamma^2(2\beta)} + \frac{i2^{2\beta} \Gamma(\beta + i\nu) e^{i\pi\beta} \Gamma(-2\beta)}{\Gamma(2\beta) \Gamma(1 - \beta + i\nu)}, & \text{for } \beta \in (0, 1/2); \\ \frac{i2^{2\beta-1} |\Gamma(\beta + i\nu)|^2 e^{\pm \pi \nu} e^{(1 \mp 1)i\pi\beta}}{\beta \Gamma^2(2\beta)}, & \text{for } \beta \geq 1/2; \\ \frac{i\kappa(\kappa - \beta) 2^{2\beta-1} \Gamma(\beta + i\nu) e^{i\pi\beta}}{\nu \Gamma(2\beta)} \left( \frac{\Gamma(\beta - i\nu) e^{\pm \pi \nu \mp i\pi\beta}}{\beta \Gamma(2\beta)} + \frac{2\Gamma(-2\beta)}{\Gamma(1 - \beta + i\nu)} \right), & \text{for } \beta \in i\mathbb{R}_+; \\ \Gamma(1 - i\nu) \Gamma(i\nu) e^{\pm \pi \nu} + \ln 2 + 2\gamma + \psi(1 + i\nu) + \frac{i\pi}{2} + \frac{i}{2\nu}, & \text{for } \beta = 0 \end{cases} \quad (4.19)$$

and

$$c_-^{\nu, \kappa} := \begin{cases} \frac{i2^{2\beta} \Gamma(\beta + i\nu) e^{i\pi\beta} \Gamma(-2\beta)}{\Gamma(2\beta) \Gamma(1 - \beta + i\nu)}, & \text{for } \beta \in (0, 1/2); \\ 0, & \text{for } \beta \geq 1/2; \\ \frac{i\kappa(\kappa - \beta) 2^{2\beta} \Gamma(\beta + i\nu) \Gamma(-2\beta) e^{i\pi\beta}}{\nu \Gamma(2\beta) \Gamma(1 - \beta + i\nu)}, & \text{for } \beta \in i\mathbb{R}_+; \\ \ln 2 + 2\gamma + \psi(1 + i\nu) + \frac{i\pi}{2} + \frac{i}{2\nu}, & \text{for } \beta = 0. \end{cases} \quad (4.20)$$

Then

$$\Phi_\infty^{\nu, \kappa}(\lambda; r) := \begin{cases} \Phi_U^{\nu, \kappa}(\lambda r) + c^{\nu, \kappa}(\lambda) \Phi_M^{\nu, \kappa}(\lambda r), & \text{for } \kappa \neq 0, \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+); \\ \Phi_U^{\nu, 0}(\lambda r), & \text{for } \kappa = 0, \operatorname{Im} \lambda > 0, \operatorname{Re} \lambda \neq 0; \\ \Phi_M^{\nu, 0}(\lambda r), & \text{for } \kappa = 0, \operatorname{Im} \lambda < 0 \end{cases} \quad (4.21)$$

is the unique (up to a constant factor) non-trivial solution of (4.1) which is square integrable at infinity.

*Proof.* According to 13.7.2 and 13.2.4 in [1], the following asymptotics hold true for large  $r$  provided  $\arg \lambda \in (-3\pi/2, \pi/2)$ :

$$M(a, b, 2i\lambda r) = (1 + O(r^{-1})) \begin{cases} \frac{\Gamma(b)e^{\pm i\pi a}(2i\lambda r)^{-a}}{\Gamma(b-a)}, & \text{for } \pm \operatorname{Re} \lambda > 0, \operatorname{Im} \lambda > 0; \\ \frac{\Gamma(b)e^{2i\lambda r}(2i\lambda r)^{a-b}}{\Gamma(a)}, & \text{for } \operatorname{Im} \lambda < 0, \end{cases}$$

unless  $a \in -\mathbb{N}_0$  or  $a - b \in \mathbb{N}_0$  (which cases are not relevant for our calculation). By 13.7.3 in [1],  $U(a, b, 2i\lambda r) = (2i\lambda r)^{-a}(1 + O(r^{-1}))$  for all  $a, b \in \mathbb{C}$ ,  $\lambda \in \mathbb{C} \setminus \overline{i\mathbb{R}_+}$  and  $r \gg 1$ . This allows us to compute the asymptotics of  $\Phi_M^{\nu, \kappa}(\lambda \cdot)$  and  $\Phi_U^{\nu, \kappa}(\lambda \cdot)$  with  $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+)$  for large positive values of the argument. Since (1.1) is in the limit point case at infinity (see the proof of Theorem 1), by Theorem 1.4 in [20] for  $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+)$  there exists a unique (up to a factor) solution  $\Phi_\infty^{\nu, \kappa}(\lambda; \cdot)$  to (4.1) which is square integrable at infinity. To construct such a solution it is enough to find a non-trivial linear combination of  $\Phi_M^{\nu, \kappa}(\lambda \cdot)$  and  $\Phi_U^{\nu, \kappa}(\lambda \cdot)$  for which the coefficient at the non square integrable asymptotic term vanishes. The coefficients in the statement of the lemma are chosen to satisfy this condition.  $\square$

We fix a solution to the spectral equation (4.1) satisfying the boundary condition (2.1) or (2.2):

**Lemma 10.** *For  $(\nu, \kappa, \theta) \in \mathfrak{M}$  every solution of (4.1) satisfying the boundary condition (2.1) (for  $\beta \in \overline{\mathbb{R}_+}$  and  $\kappa \neq 0$ ) or (2.2) (for  $\beta \in i\mathbb{R}_+$  or  $\kappa = 0$ ) with  $\lambda \in \mathbb{C} \setminus \overline{i\mathbb{R}_+}$  is proportional to*

$$\Phi_{0, \theta}^{\nu, \kappa}(\lambda; r) := a_\theta^{\nu, \kappa}(\lambda) \Phi_U^{\nu, \kappa}(\lambda r) + b_\theta^{\nu, \kappa}(\lambda) \Phi_M^{\nu, \kappa}(\lambda r), \quad (4.22)$$

with

$$a_\theta^{\nu, \kappa}(\lambda) := \begin{cases} \lambda^\beta \cos \theta, & \text{for } \beta \in \overline{\mathbb{R}_+} \text{ and } \kappa \neq 0; \\ \lambda^\beta e^{i\theta}, & \text{for } \beta \in i\mathbb{R}_+ \text{ and } \kappa \neq 0; \\ \lambda^{-i\nu} e^{i\theta}, & \text{for } \kappa = 0 \end{cases} \quad (4.23)$$

and

$$b_\theta^{\nu, \kappa}(\lambda) := \begin{cases} \lambda^{-\beta} \sin \theta, & \text{for } \beta \in \mathbb{R}_+; \\ \sin \theta - \cos \theta \ln \lambda, & \text{for } \beta = 0 \text{ and } \kappa \neq 0; \\ \lambda^{-\beta} e^{-i\theta}, & \text{for } \beta \in i\mathbb{R}_+ \text{ and } \kappa \neq 0; \\ \lambda^{i\nu} e^{-i\theta}, & \text{for } \kappa = 0. \end{cases} \quad (4.24)$$

Here the analytic branches of powers and logarithms of  $\lambda$  are fixed by the convention  $\arg \lambda \in (-3\pi/2, \pi/2)$ . For  $\lambda \in \mathbb{R} \setminus \{0\}$  both components of  $\Phi_{0, \theta}^{\nu, \kappa}$  are real-valued.

*Proof.* By the last statement of Lemma 7 any solution to (4.1) is a linear combination of  $\Phi_U^{\nu, \kappa}(\lambda \cdot)$  and  $\Phi_M^{\nu, \kappa}(\lambda \cdot)$ . For  $\beta \geq 1/2$  (hence  $\theta = \pi/2$ ), the only admissible solutions are multiples of  $\Phi_M^{\nu, \kappa}(\lambda \cdot)$ . Otherwise, substituting the general solution into (2.1) (or (2.2), respectively), we conclude the statement of the lemma from the asymptotics (4.16), (4.17) and (3.1), (3.2).  $\square$

**Lemma 11.** For any  $(\nu, \kappa, \theta) \in \mathfrak{M}$  the operator  $D_\theta^{\nu, \kappa}$  has no eigenvalues.

*Proof.* Assume that  $\lambda \in \mathbb{R} \setminus \{0\}$  is an eigenvalue of  $D_\theta^{\nu, \kappa}$ . Then the corresponding eigenfunction must be a linear combination of  $\Phi_M^{\nu, \kappa}(\lambda \cdot)$  and  $\Phi_U^{\nu, \kappa}(\lambda \cdot)$  by Lemma 7. No such non-trivial linear combination, however, can be square integrable, as follows from analysis of asymptotics at infinity similar to the one from the proof of Lemma 9. Analogously, for  $\lambda = 0$  any solution of (4.1) is a linear combination of (3.1) and (3.2) which are not square integrable at infinity.  $\square$

**Lemma 12.** Let  $(\nu, \kappa, \theta) \in \mathfrak{M}$ . For any two solutions  $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ ,  $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$  of (4.1) the function

$$W[f, g](r) := f_1(r)g_2(r) - f_2(r)g_1(r) \quad (4.25)$$

does not depend on  $r \in \mathbb{R}_+$ . Moreover, the formulae

$$\begin{aligned} W[\Phi_\infty^{\nu, \kappa}(\lambda; \cdot), \Phi_{0, \theta}^{\nu, \kappa}(\lambda; \cdot)] = \\ W[\Phi_M^{\nu, \kappa}, \Phi_U^{\nu, \kappa}] \begin{cases} c^{\nu, \kappa}(\lambda) a_\theta^{\nu, \kappa}(\lambda) - b_\theta^{\nu, \kappa}(\lambda), & \text{for } \kappa \neq 0, \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+); \\ -b_\theta^{\nu, 0}(\lambda), & \text{for } \kappa = 0, \operatorname{Im} \lambda > 0, \operatorname{Re} \lambda \neq 0; \\ a_\theta^{\nu, 0}(\lambda), & \text{for } \kappa = 0, \operatorname{Im} \lambda < 0 \end{cases} \end{aligned} \quad (4.26)$$

and

$$W[\Phi_M^{\nu, \kappa}, \Phi_U^{\nu, \kappa}] = \begin{cases} -2\beta\kappa(\kappa + \beta), & \text{for } \beta \in \mathbb{R}_+ \setminus \{-\kappa\}; \\ -\beta, & \text{for } \beta = -\kappa \in \mathbb{R}_+; \\ \nu, & \text{for } \beta = 0 \text{ and } \kappa \neq 0; \\ -2\kappa^2\nu\beta, & \text{for } \beta \in i\mathbb{R}_+ \text{ and } \kappa \neq 0; \\ 2i, & \text{for } \kappa = 0 \end{cases} \quad (4.27)$$

hold with the coefficients defined in Lemmata 9 and 10.

*Proof.* The independence of (4.25) from  $r \in \mathbb{R}_+$  follows from (4.1) by a straightforward computation. Hence the analyticity of  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$  in  $\mathbb{C} \setminus i\mathbb{R}_+$  implies that the function  $W[\Phi_M^{\nu, \kappa}, \Phi_U^{\nu, \kappa}]$  is constant on  $\mathbb{C} \setminus i\mathbb{R}_+$ . Relation (4.26) follows by substituting (4.21) and (4.22) into (4.25). At last, (4.27) follows from (4.25) by passing to the limit  $r \rightarrow +0$  and using Lemma 8.  $\square$

**Lemma 13.** Let  $(\nu, \kappa, \theta) \in \mathfrak{M}$ . For  $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+)$ , the Green function  $G_\theta^{\nu, \kappa} : \mathbb{R} \times (\mathbb{R}_+)^2 \rightarrow \mathbb{C}^{2 \times 2}$  of  $D_\theta^{\nu, \kappa}$ , i.e. the integral kernel of  $(D_\theta^{\nu, \kappa} - \lambda \mathbb{I})^{-1}$ , is given by

$$G_\theta^{\nu, \kappa}(\lambda; x, y) := \frac{1}{W[\Phi_\infty^{\nu, \kappa}(\lambda; \cdot), \Phi_{0, \theta}^{\nu, \kappa}(\lambda; \cdot)]} \begin{cases} \Phi_{0, \theta}^{\nu, \kappa}(\lambda; x) (\Phi_\infty^{\nu, \kappa}(\lambda; y))^T, & \text{for } x < y; \\ \Phi_\infty^{\nu, \kappa}(\lambda; x) (\Phi_{0, \theta}^{\nu, \kappa}(\lambda; y))^T, & \text{for } x \geq y. \end{cases} \quad (4.28)$$

*Proof.* Fix  $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}_+)$ . For  $f \in L^2(\mathbb{R}_+, \mathbb{C}^2)$  and  $x \in \mathbb{R}_+$  let

$$\begin{aligned} (R_\theta^{\nu, \kappa}(\lambda)f)(x) &:= \frac{\Phi_\infty^{\nu, \kappa}(\lambda; x)}{W[\Phi_\infty^{\nu, \kappa}(\lambda; \cdot), \Phi_{0, \theta}^{\nu, \kappa}(\lambda; \cdot)]} \int_0^x (\Phi_{0, \theta}^{\nu, \kappa}(\lambda; y))^T f(y) dy \\ &+ \frac{\Phi_{0, \theta}^{\nu, \kappa}(\lambda; x)}{W[\Phi_\infty^{\nu, \kappa}(\lambda; \cdot), \Phi_{0, \theta}^{\nu, \kappa}(\lambda; \cdot)]} \int_x^\infty (\Phi_\infty^{\nu, \kappa}(\lambda; y))^T f(y) dy. \end{aligned} \quad (4.29)$$

Since  $\Phi_{0,\theta}^{\nu,\kappa}(\lambda; \cdot) \in \mathcal{L}^2((0, x))$  and  $\Phi_\infty^{\nu,\kappa}(\lambda; \cdot) \in \mathcal{L}^2((x, \infty))$ , the integrals on the right hand side of (4.29) converge and the map  $f \mapsto (R_\theta^{\nu,\kappa}(\lambda)f)(x)$  is continuous on  $\mathcal{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ . It is easy to observe that for all  $f \in \mathcal{C}_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$  the function  $R_\theta^{\nu,\kappa}(\lambda)f$  belongs to the domain of  $D_\theta^{\nu,\kappa}$  and satisfies

$$(D_\theta^{\nu,\kappa} - \lambda \mathbb{I})R_\theta^{\nu,\kappa}(\lambda)f = f.$$

Now for arbitrary  $f \in \mathcal{L}^2(\mathbb{R}_+, \mathbb{C}^2)$  let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of  $\mathcal{C}_0^\infty(\mathbb{R}_+, \mathbb{C}^2)$ -functions converging to  $f$  in  $\mathcal{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ . Applying (4.29) to  $f_n$  and passing to the limit  $n \rightarrow \infty$  we observe that  $R_\theta^{\nu,\kappa}(\lambda)f_n \xrightarrow[n \rightarrow \infty]{} R_\theta^{\nu,\kappa}(\lambda)f$  pointwise in  $\mathbb{R}_+$ . On the other hand, since  $\lambda$  belongs to the resolvent set of  $D_\theta^{\nu,\kappa}$ ,

$$R_\theta^{\nu,\kappa}(\lambda)f_n = (D_\theta^{\nu,\kappa} - \lambda \mathbb{I})^{-1}f_n \xrightarrow[n \rightarrow \infty]{\mathcal{L}^2(\mathbb{R}_+, \mathbb{C}^2)} (D_\theta^{\nu,\kappa} - \lambda \mathbb{I})^{-1}f.$$

This implies that  $(D_\theta^{\nu,\kappa} - \lambda \mathbb{I})^{-1}f = R_\theta^{\nu,\kappa}(\lambda)f = \int_{\mathbb{R}_+} G_\theta^{\nu,\kappa}(\lambda; \cdot, y)f(y)dy$ .  $\square$

**Lemma 14.** *For all  $(\nu, \kappa, \theta) \in \mathfrak{M}$  with  $\kappa \neq 0$  the following functions are continuous and have no zeros (see Lemmata 9, 10 for definitions):*

1.  $c_{+,\pm}^{\nu,\kappa}a_\theta^{\nu,\kappa}(\cdot) - b_\theta^{\nu,\kappa}(\cdot)$ , on  $(\pm\mathbb{R}_+) + i\overline{\mathbb{R}_+}$ ;
2.  $c_-^{\nu,\kappa}a_\theta^{\nu,\kappa}(\cdot) - b_\theta^{\nu,\kappa}(\cdot)$ , on  $(\mathbb{R} - i\overline{\mathbb{R}_+}) \setminus \{0\}$ .

*Proof.* By Definitions (4.23), (4.24) the functions in question are analytic in  $\mathbb{C} \setminus i\overline{\mathbb{R}_+}$ . Moreover, according to (4.18), (4.26) and (4.27) they are proportional to  $W[\Phi_\infty^{\nu,\kappa}(\lambda; \cdot), \Phi_{0,\theta}^{\nu,\kappa}(\lambda; \cdot)]$  in the interiors of  $(\pm\mathbb{R}_+) + i\overline{\mathbb{R}_+}$  and  $(\mathbb{R} - i\overline{\mathbb{R}_+}) \setminus \{0\}$ , respectively, with non-vanishing coefficients. But then they cannot vanish there, since otherwise  $\Phi_{0,\theta}^{\nu,\kappa}(\lambda; \cdot)$  would be proportional to  $\Phi_\infty^{\nu,\kappa}(\lambda; \cdot)$  and thus an eigenfunction of the self-adjoint operator  $D_\theta^{\nu,\kappa}$  with the eigenvalue  $\lambda \notin \mathbb{R}$ .

It remains to prove that there are no zeros for  $\lambda \in \mathbb{R} \setminus \{0\}$ . Recall that by our convention from Lemma 10, for any  $\gamma \in \mathbb{C}$

$$\lambda^\gamma = \begin{cases} |\lambda|^\gamma, & \text{for } \lambda > 0; \\ e^{-i\pi\gamma}|\lambda|^\gamma, & \text{for } \lambda < 0 \end{cases}$$

holds.

For  $\beta \geq \pi/2$  we have  $\theta = \pi/2$  and the statement follows immediately.

Suppose now that  $\beta \in (0, 1/2)$ . In this case it is enough to show that

$$\operatorname{Im}(e^{(\pm 1 - 1)i\pi\beta}c_{+,\pm}^{\nu,\kappa}) \neq 0 \quad \text{and} \quad \operatorname{Im}(e^{(\pm 1 - 1)i\pi\beta}c_-^{\nu,\kappa}) \neq 0. \quad (4.30)$$

This relations are implied by (4.19), (4.20) and

$$\operatorname{Im}\left(\frac{i2^{2\beta}\Gamma(\beta + i\nu)e^{\pm i\pi\beta}\Gamma(-2\beta)}{\Gamma(2\beta)\Gamma(1 - \beta + i\nu)}\right) = -\frac{2^{2\beta-1}|\Gamma(\beta + i\nu)|^2 e^{\pm\pi\nu}}{2\beta\Gamma^2(2\beta)},$$

which follows from the properties of the gamma function (see 5.5.3 in [1]).

For  $\beta = 0$  it is enough to establish that

$$\operatorname{Im}c_{+,\pm}^{\nu,\kappa} \neq \pi(1 \mp 1)/2 \quad \text{and} \quad \operatorname{Im}c_-^{\nu,\kappa} \neq \pi(1 \mp 1)/2, \quad (4.31)$$

which follows from (4.19), (4.20) and the relations

$$\begin{aligned}\Gamma(1 - i\nu)\Gamma(i\nu)e^{\pm\pi\nu} &= -i\pi \coth(\pi\nu) \mp i\pi, \\ \operatorname{Im}\left(\psi(1 + i\nu) + \frac{i}{2\nu}\right) &= \frac{\pi}{2} \coth(\pi\nu),\end{aligned}$$

see 5.5.3 and 5.4.18 in [1].

For  $\beta \in i\mathbb{R}_+$ , the claim follows from

$$|c_{+, \pm}^{\nu, \kappa}| \neq e^{(1 \mp 1)\pi i \beta} \quad \text{and} \quad |c_{-, \pm}^{\nu, \kappa}| \neq e^{(1 \mp 1)\pi i \beta},$$

implied by (4.19), (4.20) and

$$|c_{+, \pm}^{\nu, \kappa}|^2 = e^{(2 \mp 4)\pi i \beta} \frac{\sinh(\pi(\nu - i\beta))}{\sinh(\pi(\nu + i\beta))}, \quad |c_{-, \pm}^{\nu, \kappa}|^2 = e^{2\pi i \beta} \frac{\sinh(\pi(\nu + i\beta))}{\sinh(\pi(\nu - i\beta))},$$

see 5.4.3 in [1].  $\square$

**Lemma 15.** *Let  $I, J$  be compact subsets of  $\mathbb{R}_+$  and  $\mathbb{R} \setminus \{0\}$ , respectively. Then for every  $x > 0$  there exist  $C_{\theta, \pm}^{\nu, \kappa}(I, J; x) \in \mathbb{R}_+$  such that*

$$\sup_{(\lambda, y) \in (J \pm i(0, 1]) \times I} \|G_{\theta}^{\nu, \kappa}(\lambda; x, y)\|_{\mathbb{C}^{2 \times 2}} \leq C_{\theta, \pm}^{\nu, \kappa}(I, J; x)$$

holds.

*Proof.* The functions  $\Phi_M^{\nu, \kappa}$  and  $\Phi_U^{\nu, \kappa}$  are analytic in  $\mathbb{C} \setminus i\overline{\mathbb{R}_+}$ . Hence by (4.28), (4.26), (4.27), and Lemmata 9, 10 and 14 for every  $x > 0$  the function  $G_{\theta}^{\nu, \kappa}(\cdot; x, \cdot)$  allows a unique continuous extension from  $(J \pm i(0, 1]) \times I$  to the compact  $(J \pm i[0, 1]) \times I$ . The statement of the lemma follows.  $\square$

**Lemma 16.** *Consider the map  $E_{\theta}^{\nu, \kappa} : \mathbb{R} \times \mathbb{R}_+^2 \rightarrow \mathbb{C}^{2 \times 2}$ ,*

$$E_{\theta}^{\nu, \kappa}(\lambda; x, y) := \frac{1}{2\pi i} (G_{\theta}^{\nu, \kappa}(\lambda + i0; x, y) - G_{\theta}^{\nu, \kappa}(\lambda - i0; x, y)).$$

Then

$$E_{\theta}^{\nu, \kappa}(\lambda; x, y) = m_{\theta}^{\nu, \kappa}(\lambda) \Phi_{0, \theta}^{\nu, \kappa}(\lambda; x) (\Phi_{0, \theta}^{\nu, \kappa}(\lambda; y))^{\top} \quad (4.32)$$

holds with

$$\begin{aligned}m_{\theta}^{\nu, \kappa}(\lambda) &:= \\ &\frac{(c^{\nu, \kappa}(\lambda - i0) - c^{\nu, \kappa}(\lambda + i0))}{2\pi i W[\Phi_M^{\nu, \kappa}, \Phi_U^{\nu, \kappa}](c^{\nu, \kappa}(\lambda + i0)a_{\theta}^{\nu, \kappa}(\lambda) - b_{\theta}^{\nu, \kappa}(\lambda))(c^{\nu, \kappa}(\lambda - i0)a_{\theta}^{\nu, \kappa}(\lambda) - b_{\theta}^{\nu, \kappa}(\lambda))},\end{aligned} \quad (4.33)$$

for  $\kappa \neq 0$  and

$$m_{\theta}^{\nu, 0}(\lambda) := (4\pi)^{-1}. \quad (4.34)$$

For any bounded interval  $\mathcal{I} \subset \mathbb{R} \setminus \{0\}$  the spectral projection on  $\mathcal{I}$  of  $D_{\theta}^{\nu, \kappa}$  is given by

$$P_{\mathcal{I}}(D_{\theta}^{\nu, \kappa})f = \int_{\mathcal{I}} \int_0^{\infty} E_{\theta}^{\nu, \kappa}(\lambda; \cdot, y) f(y) dy d\lambda \quad (4.35)$$

for every compactly supported  $f \in \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ .



*Proof.* The representations (4.32) and (4.33) follow from Lemma 13 together with (4.26), (4.22) and (4.21). By Lemma 11  $D_\theta^{\nu,\kappa}$  has no eigenvalues. Hence for any compactly supported  $f \in \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$  by the Stone's formula (see e.g. Theorem 4.3 in [16]) and Lemma 13 we have

$$P_{\mathcal{I}}(D_\theta^{\nu,\kappa})f = \lim_{\varepsilon \rightarrow +0} \int_{\mathcal{I}} \int_0^\infty \frac{G_\theta^{\nu,\kappa}(\lambda + i\varepsilon; \cdot, y) - G_\theta^{\nu,\kappa}(\lambda - i\varepsilon; \cdot, y)}{2\pi i} f(y) dy d\lambda.$$

By Lemma 15 and dominated convergence we can interchange the limit and the integration obtaining (4.35).  $\square$

*Proof of Theorem 2.* For  $n \in \mathbb{N}$  let  $\mathcal{E}_n := (-n, -n^{-1}) \cup (n^{-1}, n)$ . For compactly supported  $f \in \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$  we observe that by Lemma 11, Fubini's theorem, (4.35) and (4.32)

$$\begin{aligned} \|f\|_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)}^2 &= \lim_{n \rightarrow \infty} \|P_{\mathcal{E}_n}(D_\theta^{\nu,\kappa})f\|_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)}^2 = \lim_{n \rightarrow \infty} \langle f, P_{\mathcal{E}_n}(D_\theta^{\nu,\kappa})f \rangle_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)} \\ &= \left\| \sqrt{m_\theta^{\nu,\kappa}(\cdot)} \int_0^\infty (\Phi_{0,\theta}^{\nu,\kappa}(\cdot; y))^\top f(y) dy \right\|_{\mathbb{L}^2(\mathbb{R}, \mathbb{C})}^2 \end{aligned}$$

holds. Thus  $\mathcal{U}_\theta^{\nu,\kappa}$  is well defined by (2.3) and is isometric, i.e.  $(\mathcal{U}_\theta^{\nu,\kappa})^* \mathcal{U}_\theta^{\nu,\kappa} = \mathbb{I}_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)}$ .

Now for any  $f \in \mathfrak{D}(D_\theta^{\nu,\kappa})$  integrating by parts with the help of Theorem 1 we obtain

$$\begin{aligned} \mathcal{U}_\theta^{\nu,\kappa} D_\theta^{\nu,\kappa} f &= \mathbb{L}^2\text{-}\lim_{R \rightarrow \infty} \left( \sqrt{m_\theta^{\nu,\kappa}(\cdot)} \int_{1/R}^R (\Phi_{0,\theta}^{\nu,\kappa}(\cdot; y))^\top D_\theta^{\nu,\kappa} f(y) dy \right) \\ &= \mathbb{L}^2\text{-}\lim_{R \rightarrow \infty} \left( (\cdot) \sqrt{m_\theta^{\nu,\kappa}(\cdot)} \int_{1/R}^R (\Phi_{0,\theta}^{\nu,\kappa}(\cdot; y))^\top f(y) dy \right) = \Lambda \mathcal{U}_\theta^{\nu,\kappa} f, \end{aligned} \tag{4.36}$$

hence (2.4).

It remains to prove that  $\mathcal{U}_\theta^{\nu,\kappa}$  is surjective. We denote by  $P_\theta^{\nu,\kappa}$  the orthogonal projector in  $\mathbb{L}^2(\mathbb{R}, \mathbb{C})$  onto the range of  $\mathcal{U}_\theta^{\nu,\kappa}$ . According to (4.36) the operator

$$\Lambda_\theta^{\nu,\kappa} : \mathcal{U}_\theta^{\nu,\kappa} \mathfrak{D}(D_\theta^{\nu,\kappa}) \rightarrow P_\theta^{\nu,\kappa} \mathbb{L}^2(\mathbb{R}, \mathbb{C}); \quad \Lambda_\theta^{\nu,\kappa} g := \Lambda g$$

is well-defined and self-adjoint in  $P_\theta^{\nu,\kappa} \mathbb{L}^2(\mathbb{R}, \mathbb{C})$ . Since for every  $g \in \mathfrak{D}(\Lambda)$  and  $h \in \mathfrak{D}(\Lambda_\theta^{\nu,\kappa})$

$$\langle \Lambda_\theta^{\nu,\kappa} h, P_\theta^{\nu,\kappa} g \rangle_{\mathbb{L}^2(\mathbb{R}, \mathbb{C})} = \langle h, \Lambda g \rangle_{\mathbb{L}^2(\mathbb{R}, \mathbb{C})}$$

holds, we conclude that  $P_\theta^{\nu,\kappa} \mathfrak{D}(\Lambda) \subset \mathfrak{D}((\Lambda_\theta^{\nu,\kappa})^*) = \mathfrak{D}(\Lambda_\theta^{\nu,\kappa})$  and  $\Lambda P_\theta^{\nu,\kappa} \mathfrak{D}(\Lambda) \subset P_\theta^{\nu,\kappa} \mathbb{L}^2(\mathbb{R}, \mathbb{C})$ , i.e., the range of  $\mathcal{U}_\theta^{\nu,\kappa}$  is a reducing subspace of  $\Lambda$ . Hence there exists a measurable  $\Omega \subset \mathbb{R}$  such that

$$\mathcal{U}_\theta^{\nu,\kappa} \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2) = \text{ran}(\mathbb{1}_{\mathbb{R} \setminus \Omega}(\Lambda)) \tag{4.37}$$

(combine Corollary 4.6 and Theorem 4.8 in [16]).

Our goal is to show that  $\Omega$  is a Lebesgue null set. If this is not the case, then there exists  $\delta > 0$  so that  $\Omega \cap ([-\delta^{-1}, -\delta] \cup [\delta, \delta^{-1}])$  is of positive Lebesgue measure. Since  $\Phi_{0,\theta}^{\nu,\kappa}(\cdot; 1)$  is analytic in  $\mathbb{C} \setminus i\overline{\mathbb{R}_+}$  and real-valued on  $\mathbb{R} \setminus \{0\}$ , the set

$$\Xi_0 := \{\lambda \in \mathbb{R} \setminus \{0\} : (\Phi_{0,\theta}^{\nu,\kappa})_1(\lambda; 1) = 0\}$$

where the first component of  $\Phi_{0,\theta}^{\nu,\kappa}$  vanishes is Lebesgue null, and thus at least one of the sets

$$\Xi_{\pm} := \{\lambda \in \mathbb{R} \setminus \{0\} : \pm(\Phi_{0,\theta}^{\nu,\kappa})_1(\lambda; 1) > 0\} \cap \Omega \cap ([-\delta^{-1}, -\delta] \cup [\delta, \delta^{-1}]) \quad (4.38)$$

has positive Lebesgue measure. Without restriction suppose that  $\Xi_+$  is not null. A simple calculation gives

$$(\mathcal{U}_{\theta}^{\nu,\kappa})^* g = \mathbb{L}^2\text{-}\lim_{R \rightarrow \infty} \int_{-R}^R \Phi_{0,\theta}^{\nu,\kappa}(\lambda; \cdot) g(\lambda) \sqrt{m_{\theta}^{\nu,\kappa}(\lambda)} d\lambda \quad (4.39)$$

for any  $g \in \mathbb{L}^2(\mathbb{R}, \mathbb{C})$ . Choosing  $g := \mathbb{1}_{\Xi_+}$  and using  $m_{\theta}^{\nu,\kappa}(\cdot) > 0$  we obtain

$$((\mathcal{U}_{\theta}^{\nu,\kappa})^* \mathbb{1}_{\Xi_+})_1(1) > 0. \quad (4.40)$$

By dominated convergence (4.39) implies that the map  $r \mapsto ((\mathcal{U}_{\theta}^{\nu,\kappa})^* \mathbb{1}_{\Xi_+})_1(r)$  is continuous at  $r = 1$ , thus by (4.40) we get  $(\mathcal{U}_{\theta}^{\nu,\kappa})^* \mathbb{1}_{\Xi_+} \neq 0$  in  $\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ . This is, however, not possible, since by (4.38)  $\mathbb{1}_{\Xi_+} \in (\mathcal{U}_{\theta}^{\nu,\kappa} \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2))^{\perp} = \ker((\mathcal{U}_{\theta}^{\nu,\kappa})^*)$ . The contradiction implies that  $\Omega$  is null, thus the right hand side of (4.37) coincides with  $\mathbb{L}^2(\mathbb{R}, \mathbb{C})$ .  $\square$

## 5 Proof of Theorem 3

*Proof of Theorem 3, part I.* We use  $\varphi_n := (\mathcal{U}_{\theta}^{\nu,\kappa})^*((\cdot)^{-1} \mathbb{1}_{[1/n^2, 1/n]}(\cdot))$  for  $n \in \mathbb{N}$  big enough as a test function. Theorem 2 implies that  $\varphi_n \in P_{\theta,\infty}^{\nu,\kappa} \mathfrak{D}(D_{\theta}^{\nu,\kappa}) \subset P_{\theta,\infty}^{\nu,\kappa} \mathfrak{D}(|D_{\theta}^{\nu,\kappa}|^{1/2})$  and

$$\begin{aligned} & \langle |D_{\theta}^{\nu,\kappa}|^{1/2} \varphi_n, |D_{\theta}^{\nu,\kappa}|^{1/2} \varphi_n \rangle_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)} \\ &= \langle \mathcal{U}_{\theta}^{\nu,\kappa} \varphi_n, (\mathcal{U}_{\theta}^{\nu,\kappa} D_{\theta}^{\nu,\kappa} (\mathcal{U}_{\theta}^{\nu,\kappa})^*) \mathcal{U}_{\theta}^{\nu,\kappa} \varphi_n \rangle_{\mathbb{L}^2(\mathbb{R}, \mathbb{C}^2)} = \int_{1/n^2}^{1/n} \lambda^{-1} d\lambda = \ln n. \end{aligned} \quad (5.1)$$

We also have by (4.39)

$$\begin{aligned} \langle \varphi_n, V \varphi_n \rangle_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)} &= \int_0^{\infty} \int_{1/n^2}^{1/n} \int_{1/n^2}^{1/n} \frac{\sqrt{m_{\theta}^{\nu,\kappa}(\lambda)}}{\lambda} \frac{\sqrt{m_{\theta}^{\nu,\kappa}(\mu)}}{\mu} \\ &\quad \times \langle \Phi_{0,\theta}^{\nu,\kappa}(\lambda; r), V(r) \Phi_{0,\theta}^{\nu,\kappa}(\mu; r) \rangle_{\mathbb{C}^2} d\mu d\lambda dr. \end{aligned} \quad (5.2)$$

Recalling Lemma 10 together with (4.16) and (4.17) and taking into account the boundedness of  $\Phi_U^{\nu,\kappa}$  and  $\Phi_M^{\nu,\kappa}$  on  $[1, \infty)$  for all  $\lambda, r \in \mathbb{R}_+$  (see Lemma 7 together with 13.7.2 and 13.7.3 in [1]) we obtain the decomposition

$$\Phi_{0,\theta}^{\nu,\kappa}(\lambda; r) = A_{\theta}^{\nu,\kappa}(r) + B_{\theta}^{\nu,\kappa}(\lambda; r),$$

where  $A_{\theta}^{\nu,\kappa}$  is defined in (2.5) and

$$\|B_{\theta}^{\nu,\kappa}(\lambda; r)\|_{\mathbb{C}^2} \leq C^{\nu,\kappa} \lambda r^{1-\text{Re } \beta} \quad (5.3)$$

with some finite  $C^{\nu,\kappa} > 0$ . Thus writing  $V = |V|^{1/2}(\text{sign } V)|V|^{1/2}$  and using the Cauchy inequality we can estimate

$$\begin{aligned} & \left| \langle \Phi_{0,\theta}^{\nu,\kappa}(\lambda; r), V(r) \Phi_{0,\theta}^{\nu,\kappa}(\mu; r) \rangle_{\mathbb{C}^2} \right| \\ & \leq 2 \langle A_{\theta}^{\nu,\kappa}(r), |V(r)| A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} + (C^{\nu,\kappa})^2 (\lambda^2 + \mu^2) \|V(r)\|_{\mathbb{C}^{2 \times 2}} r^{2-2 \operatorname{Re} \beta}, \end{aligned}$$

where the right hand side is integrable in  $r$  over  $\mathbb{R}_+$  by (2.6). Hence the order of integrations in (5.2) can be interchanged by Fubini's theorem. By Schwarz inequality and (5.3) we have

$$\begin{aligned} & \operatorname{Re} \int_0^\infty \langle \Phi_{0,\theta}^{\nu,\kappa}(\lambda; r), V(r) \Phi_{0,\theta}^{\nu,\kappa}(\mu; r) \rangle_{\mathbb{C}^2} dr \\ & = \operatorname{Re} \langle |V|^{1/2} (A_{\theta}^{\nu,\kappa} + B_{\theta}^{\nu,\kappa}(\lambda; \cdot)), (\text{sign } V) |V|^{1/2} (A_{\theta}^{\nu,\kappa} + B_{\theta}^{\nu,\kappa}(\mu; \cdot)) \rangle_{L^2(\mathbb{R}_+, \mathbb{C}^2)} \\ & \geq \frac{1}{2} \int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), V(r) A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr \\ & \quad - \left( \frac{\int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), |V(r)| A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr}{\int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), V(r) A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr} + \frac{1}{2} \right) \\ & \quad \times (C^{\nu,\kappa})^2 (\lambda^2 + \mu^2) \int_0^\infty \|V(r)\|_{\mathbb{C}^{2 \times 2}} r^{2-2 \operatorname{Re} \beta} dr. \end{aligned} \tag{5.4}$$

Inserting (5.4) into (5.2) we arrive at

$$\begin{aligned} \langle \varphi_n, V \varphi_n \rangle_{L^2(\mathbb{R}_+, \mathbb{C}^2)} & \geq \frac{1}{2} \int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), V(r) A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr \left( \int_{1/n^2}^{1/n} \frac{\sqrt{m_{\theta}^{\nu,\kappa}(\lambda)}}{\lambda} d\lambda \right)^2 \\ & \quad - \left( \frac{2 \int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), |V(r)| A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr}{\int_0^\infty \langle A_{\theta}^{\nu,\kappa}(r), V(r) A_{\theta}^{\nu,\kappa}(r) \rangle_{\mathbb{C}^2} dr} + 1 \right) (C^{\nu,\kappa})^2 \\ & \quad \times \int_0^\infty \|V(r)\|_{\mathbb{C}^{2 \times 2}} r^{2-2 \operatorname{Re} \beta} dr \left( \int_{1/n^2}^{1/n} \lambda \sqrt{m_{\theta}^{\nu,\kappa}(\lambda)} d\lambda \right) \left( \int_{1/n^2}^{1/n} \frac{\sqrt{m_{\theta}^{\nu,\kappa}(\lambda)}}{\lambda} d\lambda \right), \end{aligned} \tag{5.5}$$

which is positive for  $n$  big enough due to (2.6) and (1.3). It follows from (4.33), (4.34), (4.27) and Lemmata 9 and 10, that there exists  $h^{\nu,\kappa} > 0$  such that for  $(\nu, \kappa, \theta) \in \mathfrak{M}_I$

$$m_{\theta}^{\nu,\kappa}(\lambda) \geq h^{\nu,\kappa} \lambda^{-2 \operatorname{Re} \beta}$$

holds for all  $\lambda > 0$ . This implies

$$\int_{1/n^2}^{1/n} \frac{\sqrt{m_{\theta}^{\nu,\kappa}(\lambda)}}{\lambda} d\lambda \geq h^{\nu,\kappa} \begin{cases} \ln n, & \text{for } \beta \in \overline{i\mathbb{R}_+}; \\ \frac{n^{2\beta} - n^{\beta}}{\beta}, & \text{for } \beta \in (0, 1/2). \end{cases} \tag{5.6}$$

Now (5.1), (5.5) and (5.6) imply that the quadratic form of  $D_{\theta,\infty}^{\nu,\kappa}(V)$  computed on  $\varphi_n$  becomes negative for  $n$  big enough. The existence of negative spectrum of  $D_{\theta,\infty}^{\nu,\kappa}(V)$  follows by the minimax principle.  $\square$

For  $E \in (0, \infty]$  let  $P_{\theta, E}^{\nu, \kappa} := P_{[0, E)}(D_{\theta}^{\nu, \kappa})$  be the spectral projector of  $D_{\theta}^{\nu, \kappa}$  corresponding to the interval  $[0, E)$ . Under Hypothesis A the quadratic form

$$\mathfrak{d}_{\theta, E}^{\nu, \kappa}(V)[\cdot] := \left\| |D_{\theta}^{\nu, \kappa}|^{1/2} \cdot \right\|_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)}^2 - \langle |V|^{1/2} \cdot, (\text{sign } V)|V|^{1/2} \cdot \rangle_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)} \quad (5.7)$$

is closed and bounded from below on  $P_{\theta, E}^{\nu, \kappa} \mathfrak{D}(|D_{\theta}^{\nu, \kappa}|^{1/2})$ . We define

$$D_{\theta, E}^{\nu, \kappa}(V) := P_{\theta, E}^{\nu, \kappa}(D_{\theta}^{\nu, \kappa} - V)P_{\theta, E}^{\nu, \kappa}$$

as the self-adjoint operator in  $P_{\theta, E}^{\nu, \kappa} \mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$  corresponding to  $\mathfrak{d}_{\theta, E}^{\nu, \kappa}(V)$ .

For  $\tau > 0$  Hypothesis A implies that the Birman-Schwinger operator

$$B_{\theta, E}^{\nu, \kappa}(V, \tau) := (D_{\theta}^{\nu, \kappa} P_{\theta, E}^{\nu, \kappa} + \tau \mathbb{I})^{-1/2} P_{\theta, E}^{\nu, \kappa} V (D_{\theta}^{\nu, \kappa} P_{\theta, E}^{\nu, \kappa} + \tau \mathbb{I})^{-1/2} P_{\theta, E}^{\nu, \kappa} \quad (5.8)$$

is bounded in  $\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ .

We will use the following version of the Birman-Schwinger principle:

**Lemma 17.** *The equality*

$$\text{rank } P_{(-\infty, -\tau)}(D_{\theta, E}^{\nu, \kappa}(V)) = \text{rank } P_{(1, \infty)}(B_{\theta, E}^{\nu, \kappa}(V, \tau)) \quad (5.9)$$

holds for any  $\tau > 0$  with  $B_{\theta, E}^{\nu, \kappa}(V, \tau)$  defined in (5.8).

*Proof.* For any  $\varphi \in P_{\theta, E}^{\nu, \kappa} \mathfrak{D}(|D_{\theta}^{\nu, \kappa}|^{1/2})$ ,  $\tau > 0$  we have

$$\begin{aligned} & \mathfrak{d}_{\theta, E}^{\nu, \kappa}(V)[\varphi] + \tau \|\varphi\|_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)}^2 \\ &= \langle (D_{\theta}^{\nu, \kappa} P_{\theta, E}^{\nu, \kappa} + \tau \mathbb{I})^{1/2} \varphi, (\mathbb{I} - B_{\theta, E}^{\nu, \kappa}(\tau))(D_{\theta}^{\nu, \kappa} P_{\theta, E}^{\nu, \kappa} + \tau \mathbb{I})^{1/2} \varphi \rangle_{\mathbb{L}^2(\mathbb{R}_+, \mathbb{C}^2)}. \end{aligned}$$

The identity (5.9) follows by the minimax principle.  $\square$

**Lemma 18.** *For  $q \geq 1$  and  $\tau > 0$  the estimate*

$$\begin{aligned} & \text{rank } P_{(-\infty, -\tau)}(D_{\theta, E}^{\nu, \kappa}(V)) \\ & \leq \int_0^\infty \int_0^E m_{\theta}^{\nu, \kappa}(\lambda)(\lambda + \tau)^{-q} \|V_+^{q/2}(r) \Phi_{0, \theta}^{\nu, \kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda dr \end{aligned} \quad (5.10)$$

holds. If the integral on the right hand side of (5.10) is finite with  $E := \infty$ , then Hypothesis A is satisfied for  $V := V_+$ .

*Proof.* For  $q \geq 1$  let  $\mathfrak{S}^q$  be the  $q$ th Schatten-von-Neumann class of compact operators. Then for any self-adjoint, non-negative operators  $A$  and  $B$  with  $A^q B^q \in \mathfrak{S}^2$  we have  $AB \in \mathfrak{S}^{2q}$  and  $\|AB\|_{\mathfrak{S}^{2q}}^{2q} \leq \|A^q B^q\|_{\mathfrak{S}^2}^2$  (see Appendix B in [10]). Hence by (4.39)

$$\begin{aligned} & \|V_+^{1/2} (D_{\theta}^{\nu, \kappa} P_{\theta, E}^{\nu, \kappa} + \tau \mathbb{I})^{-1/2} P_{\theta, E}^{\nu, \kappa}\|_{\mathfrak{S}^{2q}}^{2q} \\ & \leq \|V_+^{q/2} (\mathcal{U}_{\theta}^{\nu, \kappa})^* \mathcal{U}_{\theta}^{\nu, \kappa} (D_{\theta}^{\nu, \kappa} P_{\theta, E}^{\nu, \kappa} + \tau \mathbb{I})^{-q/2} P_{\theta, E}^{\nu, \kappa} (\mathcal{U}_{\theta}^{\nu, \kappa})^*\|_{\mathfrak{S}^2}^2 \\ & = \|V_+^{q/2} (\mathcal{U}_{\theta}^{\nu, \kappa})^* \mathbb{1}_{[0, E)}(\cdot)(\cdot + \tau)^{-q/2}\|_{\mathfrak{S}^2}^2 \\ & = \int_0^\infty \int_0^E m_{\theta}^{\nu, \kappa}(\lambda)(\lambda + \tau)^{-q} \|V_+^{q/2}(r) \Phi_{0, \theta}^{\nu, \kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda dr. \end{aligned} \quad (5.11)$$

Estimating the right hand side of (5.9) from above by

$$\begin{aligned} & \|(D_\theta^{\nu,\kappa} P_{\theta,E}^{\nu,\kappa} + \tau \mathbb{I})^{-1/2} P_{\theta,E}^{\nu,\kappa} V_+ (D_\theta^{\nu,\kappa} P_{\theta,E}^{\nu,\kappa} + \tau \mathbb{I})^{-1/2} P_{\theta,E}^{\nu,\kappa}\|_{\mathfrak{S}^q}^q \\ &= \|V_+^{1/2} (D_\theta^{\nu,\kappa} P_{\theta,E}^{\nu,\kappa} + \tau \mathbb{I})^{-1/2} P_{\theta,E}^{\nu,\kappa}\|_{\mathfrak{S}^{2q}}^{2q} \end{aligned}$$

and applying (5.11) we conclude (5.10).

If the integral on the right hand side of (5.10) is finite with  $E := \infty$ , then (5.11) implies that  $P_{\theta,\infty}^{\nu,\kappa} V_+ P_{\theta,\infty}^{\nu,\kappa}$  is a form compact perturbation of  $D_\theta^{\nu,\kappa}$  in  $P_{\theta,\infty}^{\nu,\kappa} \mathfrak{L}^2(\mathbb{R}_+, \mathbb{C}^2)$ . Hypothesis A follows with standard arguments.  $\square$

*Proof of Theorem 3, part II.* Applying Lemma 18 and passing to  $\tau \rightarrow +0$  in (5.10) we obtain

$$\begin{aligned} & \text{rank } P_{(-\infty,0)}(D_{\theta,\infty}^{\nu,\kappa}(V)) \\ & \leq \int_0^\infty \|V_+^q(r)\|_{\mathbb{C}^{2 \times 2}} \int_0^\infty \lambda^{-q} m_\theta^{\nu,\kappa}(\lambda) \|\Phi_{0,\theta}^{\nu,\kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda dr. \end{aligned} \quad (5.12)$$

In the case  $\beta \in (0, 1/2)$  and  $\theta \in (0, \pi) \setminus \{\pi/2\}$ , we rescale the variable  $\mu := |\cot \theta|^{1/(2\beta)} \lambda$  and observe

$$\int_0^\infty \lambda^{-q} m_\theta^{\nu,\kappa}(\lambda) \|\Phi_{0,\theta}^{\nu,\kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda \quad (5.13)$$

$$= \frac{|\cot \theta|^{(q-2\beta-1)/(2\beta)}}{\sin^2 \theta} \int_0^\infty \mu^{-q+2\beta} \tilde{m}_\theta^{\nu,\kappa}(\mu) \|\Phi_{0,\theta}^{\nu,\kappa}(|\cot \theta|^{-1/(2\beta)} \mu; r)\|_{\mathbb{C}^2}^2 d\mu, \quad (5.14)$$

where

$$\tilde{m}_\theta^{\nu,\kappa}(\mu) := \sin^2 \theta |\cot \theta| \mu^{-2\beta} m_\theta^{\nu,\kappa}(|\cot \theta|^{-1/(2\beta)} \mu)$$

satisfies

$$|\tilde{m}_\theta^{\nu,\kappa}(\mu)| \leq C^{\nu,\kappa} \begin{cases} \mu^{-4\beta}, & \text{for } \mu \geq 1; \\ 1, & \text{for } \mu \leq 1 \end{cases} \quad (5.15)$$

with some  $C^{\nu,\kappa} > 0$  independent from  $\theta$  (see (4.33), (4.23), (4.24), (4.18) and (4.30)). By Lemma 10 we get

$$\|\Phi_{0,\theta}^{\nu,\kappa}(\lambda; r)\|_{\mathbb{C}_2}^2 \leq 2 \cos^2 \theta \lambda^{2\beta} \|\Phi_U^{\nu,\kappa}(\lambda r)\|_{\mathbb{C}_2}^2 + 2 \sin^2 \theta \lambda^{-2\beta} \|\Phi_M^{\nu,\kappa}(\lambda r)\|_{\mathbb{C}_2}^2. \quad (5.16)$$

By Lemmata 7 and 8 for  $\beta > 0$  there exist finite constants  $C_M^{\nu,\kappa}$  and  $C_U^{\nu,\kappa}$  such that for any  $x \in \mathbb{R}_+$

$$\|\Phi_M^{\nu,\kappa}(x)\|_{\mathbb{C}_2}^2 \leq C_M^{\nu,\kappa} \begin{cases} x^{2\beta}, & \text{for } x \leq 1; \\ 1, & \text{for } x \geq 1; \end{cases} \quad \|\Phi_U^{\nu,\kappa}(x)\|_{\mathbb{C}_2}^2 \leq C_U^{\nu,\kappa} \begin{cases} x^{-2\beta}, & \text{for } x \leq 1; \\ 1, & \text{for } x \geq 1. \end{cases} \quad (5.17)$$

Substituting (5.16) into (5.13) and using the estimates (5.17) and (5.15) we obtain (1.4).

For  $\beta > 0$  and  $\theta = \pi/2$ , starting from (5.12), rescaling  $\lambda =: \mu/r$  and using Lemma 10 and (4.33) we conclude

$$\begin{aligned} & \text{rank } P_{(-\infty, 0)}(D_{\theta, \infty}^{\nu, \kappa}(V)) \\ & \leq m_{\pi/2}^{\nu, \kappa}(1) \int_0^\infty \mu^{-q} \|\Phi_M^{\nu, \kappa}(\mu)\|_{\mathbb{C}^2}^2 d\mu \int_0^\infty \|V_+^q(r)\|_{\mathbb{C}^{2 \times 2}} r^{q-1} dr, \end{aligned}$$

where the  $\mu$ -integral is finite by (5.17).  $\square$

We now turn to the case of  $\beta = 0$ , in which the integral on the right hand side of (5.10) does not converge for  $E = \infty$ . To remedy this problem, we use

**Lemma 19.** *Let  $\|V_+\|_\infty := \|V_+\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^{2 \times 2})}$  be finite. Then for any  $\tau \geq 0$  we have*

$$\text{rank } P_{(-\infty, -\tau)}(D_{\theta, \infty}^{\nu, \kappa}(V)) \leq \text{rank } P_{(-\infty, -\tau)}(D_{\theta, 2\|V_+\|_\infty}^{\nu, \kappa}(2V_+)).$$

*Proof.* For any  $\varphi \in P_{\theta, \infty}^{\nu, \kappa}(\mathfrak{D}(|D_\theta^{\nu, \kappa}|^{1/2}))$  let  $\check{\varphi} := P_{\theta, 2\|V_+\|_\infty}^{\nu, \kappa} \varphi$ ;  $\hat{\varphi} := \varphi - \check{\varphi}$ . Then (5.7) implies

$$\begin{aligned} & \mathfrak{d}_{\theta, \infty}^{\nu, \kappa}(V)[\varphi] \geq \mathfrak{d}_{\theta, \infty}^{\nu, \kappa}(V_+)[\check{\varphi}] + \mathfrak{d}_{\theta, \infty}^{\nu, \kappa}(V_+)[\hat{\varphi}] - 2 \text{Re} \langle V_+^{1/2} \check{\varphi}, V_+^{1/2} \hat{\varphi} \rangle_{L^2(\mathbb{R}_+, \mathbb{C}^2)} \\ & \geq \mathfrak{d}_{\theta, 2\|V_+\|_\infty}^{\nu, \kappa}(2V_+)[\check{\varphi}] + \left( \| |D_\theta^{\nu, \kappa}|^{1/2} \hat{\varphi} \|_{L^2(\mathbb{R}_+, \mathbb{C}^2)}^2 - 2 \| V_+^{1/2} \hat{\varphi} \|_{L^2(\mathbb{R}_+, \mathbb{C}^2)}^2 \right). \end{aligned} \quad (5.18)$$

Since  $\| |D_\theta^{\nu, \kappa}|^{1/2} \hat{\varphi} \|_{L^2(\mathbb{R}_+, \mathbb{C}^2)}^2 \geq 2 \| V_+ \|_\infty \| \hat{\varphi} \|_{L^2(\mathbb{R}_+, \mathbb{C}^2)}^2$ , the term in the parenthesis on the right hand side of (5.18) is non-negative. The statement of the lemma now follows from the minimax principle.  $\square$

*Proof of Theorem 3, part III.* Combining Lemma 19 with Lemma 18 (with  $\tau \rightarrow +0$ ,  $q := 1$ ) we obtain

$$\begin{aligned} & \text{rank } P_{(-\infty, 0)}(D_{\theta, \infty}^{\nu, \kappa}(V)) \\ & \leq \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}} \int_0^{2\|V_+\|_\infty} \lambda^{-1} m_\theta^{\nu, \kappa}(\lambda) \|\Phi_{0, \theta}^{\nu, \kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda dr. \end{aligned} \quad (5.19)$$

Inserting the definitions (4.33), (4.22), (4.23) and (4.24) and performing the rescaling  $\lambda =: e^{\tan \theta} \mu$  we rewrite the inner integral on the right hand side of (5.19) as

$$\begin{aligned} & \int_0^{2\|V_+\|_\infty} \lambda^{-1} m_\theta^{\nu, \kappa}(\lambda) \|\Phi_{0, \theta}^{\nu, \kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda \\ & = \int_0^{2\|V_+\|_\infty e^{-\tan \theta}} \mu^{-1} m_0^{\nu, \kappa}(\mu) \|f^{\nu, \kappa}(\mu e^{\tan \theta} r) + g^{\nu, \kappa}(\mu; e^{\tan \theta} r)\|_{\mathbb{C}^2}^2 d\mu \end{aligned} \quad (5.20)$$

with  $f^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \mathbb{C}^2$  and  $g^{\nu, \kappa} : \mathbb{R}_+^2 \rightarrow \mathbb{C}^2$  given by

$$f^{\nu, \kappa}(x) := \Phi_U^{\nu, \kappa}(x) - \ln(x) \Phi_M^{\nu, \kappa}(x), \quad g^{\nu, \kappa}(\zeta, x) := \ln(x) \Phi_M^{\nu, \kappa}(\zeta x).$$

Lemma 8 and boundedness of  $\Phi_U^{\nu, \kappa}$  on  $[1, \infty)$  and  $\Phi_M^{\nu, \kappa}$  on  $\mathbb{R}_+$  imply the estimates

$$\|f^{\nu, \kappa}(\cdot)\|_{\mathbb{C}^2}^2 \leq C_f^{\nu, \kappa} \ln^2(e + \cdot), \quad \|g^{\nu, \kappa}(\zeta, \cdot)\|_{\mathbb{C}^2}^2 \leq C_g^{\nu, \kappa} \ln^2(\cdot)$$

with finite constants  $C_f^{\nu,\kappa}$  and  $C_g^{\nu,\kappa}$  independent of  $\zeta$ . Moreover, (4.33) and (4.31) imply the bound

$$m_0^{\nu,\kappa}(\cdot) \leq C_m^{\nu,\kappa} (1 + \ln^2(\cdot))^{-1}.$$

Applying the above estimates to the right hand side of (5.20) and using the monotonicity of the logarithm we conclude

$$\begin{aligned} & \int_0^{2\|V_+\|_\infty} \lambda^{-1} m_\theta^{\nu,\kappa}(\lambda) \|\Phi_{0,\theta}^{\nu,\kappa}(\lambda; r)\|_{\mathbb{C}^2}^2 d\lambda \\ & \leq 2C_m^{\nu,\kappa} \left( C_f^{\nu,\kappa} \ln^2(e + 2\|V_+\|_\infty r) + C_g^{\nu,\kappa} \ln^2(e^{\tan \theta} r) \right) \int_0^{2\|V_+\|_\infty e^{-\tan \theta}} \frac{d\mu}{\mu(1 + \ln^2 \mu)}, \end{aligned}$$

where the integral can be estimated by  $\pi$ . Substituting into (5.19) we obtain (1.5).  $\square$

## 6 Proof of Theorem 4

**Lemma 20.** *For  $\tau > 0$ ,  $q > 1$ , and the combinations of  $\nu, \kappa \in \mathbb{R}$  and  $\theta \in [0, \pi)$  covered in (2.8) there exists a constant  $k^{\nu,\kappa} > 0$  such that*

$$\text{rank } P_{(-\infty, -\tau)}(D_{\theta, \infty}^{\nu,\kappa}(V)) \leq \frac{\tau^{1-q}}{q-1} k^{\nu,\kappa} \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^q W_\theta^{\nu,\kappa}(r) dr$$

holds.

*Proof.* Lemma 18 implies

$$\begin{aligned} & \text{rank } P_{(-\infty, -\tau)}(D_{\theta, \infty}^{\nu,\kappa}(V)) \\ & \leq \frac{\tau^{1-q}}{q-1} \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^q \left\| \sqrt{m_\theta^{\nu,\kappa}(\cdot)} \Phi_{0,\theta}^{\nu,\kappa}(\cdot; r) \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)}^2 dr \end{aligned}$$

holds for all  $q > 1$ . It remains to obtain the estimate

$$\left\| \sqrt{m_\theta^{\nu,\kappa}(\cdot)} \Phi_{0,\theta}^{\nu,\kappa}(\cdot; r) \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)}^2 \leq k^{\nu,\kappa} W_\theta^{\nu,\kappa}(r) \quad (6.1)$$

for all  $r > 0$  with finite  $k^{\nu,\kappa} > 0$  and  $W_\theta^{\nu,\kappa}$  defined in (2.8).

*Case  $\nu^2 = \kappa^2 \neq 0$  and  $\theta = \pi/2$ :* By (4.33) and Lemma 10  $m_{\pi/2}^{\nu,\kappa}(\lambda) = m_{\pi/2}^{\nu,\kappa}(1)$  for all  $\lambda \in \mathbb{R}_+$  and for all  $r > 0$  we have

$$\left\| \Phi_{0,\pi/2}^{\nu,\kappa}(\cdot; r) \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)} = \left\| \Phi_M^{\nu,\kappa}(\cdot r) \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)} = \left\| \Phi_M^{\nu,\kappa} \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)} < \infty.$$

*Case  $\kappa \neq 0$ ,  $\beta \in i\mathbb{R}_+$  and  $\theta \in [0, \pi)$ :* A computation based on Formula 5.4.3 in [1] allows us to justify the relations

$$c_{+,+}^{\nu,\kappa} \cdot \overline{c_{-}^{\nu,\kappa}} = 1 \quad \text{and} \quad c_{+,+}^{\nu,\kappa} (c_{-}^{\nu,\kappa})^{-1} \geq 1 \text{ for } \nu \geq 0.$$

for the coefficients introduced in (4.19), (4.20) provided  $\beta \in \mathbb{R}_+$ . Hence there exist  $\rho \in (1, \infty)$  and  $\omega \in [0, 2\pi)$  depending only on  $\kappa$  and  $\nu$  such that

$$c_{+,+}^{\nu,\kappa} = \rho^{\text{sign } \nu} e^{i\omega} \quad \text{and} \quad c_{-}^{\nu,\kappa} = \rho^{-\text{sign } \nu} e^{i\omega}.$$

Substituting this into (4.33) by (4.18), (4.27), (4.23) and (4.24) we get

$$\begin{aligned} m_{\theta}^{\nu, \kappa}(\lambda) &= \frac{1}{4\pi\kappa^2|\nu||\beta|} \cdot \frac{\rho - \rho^{-1}}{\rho + \rho^{-1} - 2\cos(\omega + 2\theta + 2|\beta|\ln\lambda)} \\ &\leq \frac{1}{4\pi\kappa^2|\beta\nu|} \cdot \frac{\rho - \rho^{-1}}{\rho + \rho^{-1} - 2} = \frac{1}{4\pi\kappa^2|\beta\nu|} \cdot \frac{\rho + 1}{\rho - 1}. \end{aligned}$$

Taking into account

$$\begin{aligned} \|\Phi_{0,\theta}^{\nu, \kappa}(\cdot; r)\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)} &= \left\| 2\operatorname{Re}((\cdot)^{-\beta} e^{-i\theta} \Phi_M^{\nu, \kappa}(\cdot r)) \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)} \\ &\leq 2\|\Phi_M^{\nu, \kappa}\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)} < \infty \end{aligned}$$

we obtain (6.1).

*Case  $\kappa = 0$ ,  $\nu \in \mathbb{R}$  and  $\theta \in [0, \pi)$ :* Here (6.1) follows immediately from (4.34), Lemma 10 and (4.5).

*Case  $\nu^2 = \kappa^2 \neq 0$  and  $\theta \in [0, \pi) \setminus \{\pi/2\}$ :* The left hand side of (6.1) coincides with  $Z_0^{\nu, \kappa}(e^{\tan\theta} r)$ , where  $Z_0^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \mathbb{R}$  is defined by

$$Z_0^{\nu, \kappa}(x) := \sup_{\zeta \in \mathbb{R}_+} \frac{|c_- - c_{+,+}| \|\Phi_U^{\nu, \kappa}(\zeta x) - (\ln \zeta) \Phi_M^{\nu, \kappa}(\zeta x)\|_{\mathbb{C}^2}^2}{|2\pi\nu(c_{+,+}^{\nu, \kappa} + \ln \zeta)(c_-^{\nu, \kappa} + \ln \zeta)|}. \quad (6.2)$$

By Lemma 14 we can estimate

$$\frac{|c_- - c_{+,+}|}{|2\pi\nu(c_{+,+}^{\nu, \kappa} + \ln \zeta)(c_-^{\nu, \kappa} + \ln \zeta)|} \leq \frac{K^{\nu, \kappa}}{1 + \ln^2 \zeta} \quad (6.3)$$

with some  $K^{\nu, \kappa} \in \mathbb{R}_+$ . On the other hand, Lemma 8 and boundedness of  $\Phi_U^{\nu, \kappa}$  and  $\Phi_M^{\nu, \kappa}$  on  $(1, \infty)$  imply

$$\|\Phi_U^{\nu, \kappa}(\zeta x) - (\ln \zeta) \Phi_M^{\nu, \kappa}(\zeta x)\|_{\mathbb{C}^2}^2 \leq L^{\nu, \kappa} \begin{cases} 1 + \ln^2 x, & \text{for } \zeta \leq x^{-1}; \\ 1 + \ln^2 \zeta, & \text{for } \zeta \geq x^{-1} \end{cases} \quad (6.4)$$

with finite  $L^{\nu, \kappa}$ . Inserting (6.3) and (6.4) into (6.2) we obtain (6.1).

*Case  $\beta \in (0, 1/2)$  and  $\theta \in (0, \pi)$  or  $\beta \geq 1/2$  and  $\theta = \pi/2$ :* For  $\theta = \pi/2$  the left hand side of (6.1) is given by  $m_{\pi/2}^{\nu, \kappa}(1) \|\Phi_M^{\nu, \kappa}\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)}^2$  and the statement follows. Otherwise, by (4.33), (4.27) and Lemma 10 we have

$$\left\| \sqrt{m_{\theta}^{\nu, \kappa}(\cdot)} \Phi_{0,\theta}^{\nu, \kappa}(\cdot; r) \right\|_{L^\infty(\mathbb{R}_+, \mathbb{C}^2)}^2 = Z_{\operatorname{sign}(\tan \theta)}^{\nu, \kappa}(|\tan \theta|^{1/(2\beta)} r), \quad (6.5)$$

where  $Z_{\pm}^{\nu, \kappa} : \mathbb{R}_+ \rightarrow \mathbb{R}$  is defined by

$$Z_{\pm}^{\nu, \kappa}(x) := \sup_{\zeta \in \mathbb{R}_+} \frac{|c_- - c_{+,+}| \|\zeta \Phi_U^{\nu, \kappa}(\zeta^{1/(2\beta)} x) \pm \Phi_M^{\nu, \kappa}(\zeta^{1/(2\beta)} x)\|_{\mathbb{C}^2}^2}{|2\pi W[\Phi_M^{\nu, \kappa}, \Phi_U^{\nu, \kappa}](c_{+,+}^{\nu, \kappa} \zeta \mp 1)(c_-^{\nu, \kappa} \zeta \mp 1)|}. \quad (6.6)$$

By (4.27) and (4.30) we can estimate

$$\frac{|c_- - c_{+,+}|}{|2\pi W[\Phi_M^{\nu, \kappa}, \Phi_U^{\nu, \kappa}](c_{+,+}^{\nu, \kappa} \zeta \mp 1)(c_-^{\nu, \kappa} \zeta \mp 1)|} \leq \frac{X^{\nu, \kappa}}{1 + \zeta^2} \quad (6.7)$$



with some finite  $X^{\nu,\kappa}$ . On the other hand, (5.17) implies

$$\begin{aligned} & \left\| \zeta \Phi_U^{\nu,\kappa}(\zeta^{1/(2\beta)}x) \pm \Phi_M^{\nu,\kappa}(\zeta^{1/(2\beta)}x) \right\|_{\mathbb{C}^2}^2 \\ & \leq Y^{\nu,\kappa} \begin{cases} \zeta(x^{-2\beta} + x^{2\beta}), & \text{for } \zeta \leq x^{-2\beta}; \\ 1 + \zeta^2, & \text{for } \zeta \geq x^{-2\beta} \end{cases} \end{aligned} \quad (6.8)$$

with finite  $Y^{\nu,\kappa}$ . Inserting (6.7) and (6.8) into (6.6) we obtain

$$\begin{aligned} Z_{\pm}^{\nu,\kappa}(x) & \leq X^{\nu,\kappa} Y^{\nu,\kappa} \max \left\{ 1, (x^{-2\beta} + x^{2\beta}) \sup_{\zeta \leq x^{-2\beta}} \frac{\zeta}{1 + \zeta^2} \right\} \\ & \leq X^{\nu,\kappa} Y^{\nu,\kappa} \max \{ 1, x^{-2\beta} \}. \end{aligned}$$

Inserting into (6.5) we get (6.1).  $\square$

*Proof of Theorem 4, part a).* The statement follows from Lemma 20 by the usual arguments (see e.g. the proof of Theorem 1.1 in [4]): Let  $\gamma > 0$  and  $q \in (1, 1 + \gamma)$ . By the minimax principle we have

$$\begin{aligned} \text{tr} (D_{\theta,\infty}^{\nu,\kappa}(V))_-^\gamma & = \int_0^\infty \gamma \tau^{\gamma-1} \text{rank} P_{(-\infty, -\tau)}(D_{\theta,\infty}^{\nu,\kappa}(V)) d\tau \\ & \leq \int_0^\infty \gamma \tau^{\gamma-1} \text{rank} P_{(-\infty, -\tau/2)}(D_{\theta,\infty}^{\nu,\kappa}((V_+ - \tau/2)_+)) d\tau \\ & \leq \frac{\gamma k^{\nu,\kappa}}{q-1} \int_0^\infty W_\theta^{\nu,\kappa}(r) \int_0^\infty \tau^{\gamma-1} \|(V_+(r) - \tau/2)_+\|_{\mathbb{C}^{2 \times 2}}^q (\tau/2)^{1-q} d\tau dr. \end{aligned} \quad (6.9)$$

Passing to the new integration variable  $\sigma := \tau \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^{-1}/2$  in the inner integral we can rewrite the right hand side of (6.9) as

$$2^\gamma \gamma k^{\nu,\kappa} \left( \frac{1}{q-1} \int_0^1 (1-\sigma)^q \sigma^{\gamma-q} d\sigma \right) \int_0^\infty \|V_+(r)\|_{\mathbb{C}^{2 \times 2}}^{1+\gamma} W_\theta^{\nu,\kappa}(r) dr.$$

Minimising in  $q \in (1, 1 + \gamma)$  we arrive at (1.6).  $\square$

*Proof of Theorem 4, part b).* Let  $q \in (1, 1 + \gamma - 2\beta)$ . Applying Lemma 18 and the minimax principle we get by (4.33) and Lemma 10

$$\begin{aligned} \text{tr} (D_{0,\infty}^{\nu,\kappa}(V))_-^\gamma & = \int_0^\infty \gamma \tau^{\gamma-1} \text{rank} P_{(-\infty, -\tau)}(D_{0,\infty}^{\nu,\kappa}(V)) d\tau \\ & \leq \int_0^\infty \gamma \tau^{\gamma-1} \text{rank} P_{(-\infty, -\tau/2)}(D_{0,\infty}^{\nu,\kappa}((V_+ - \tau/2)_+)) d\tau \\ & \leq \int_0^\infty \int_0^{2\|V_+(r)\|_{\mathbb{C}^{2 \times 2}}} \gamma \tau^{\gamma-1} m_0^{\nu,\kappa}(1) \left( \|V_+(r)\|_{\mathbb{C}^{2 \times 2}} - \frac{\tau}{2} \right)^q \left( \frac{\tau}{2} \right)^{1-q} \Omega_q^{\nu,\kappa} \left( \frac{\tau r}{2} \right) d\tau dr, \end{aligned} \quad (6.10)$$

with  $\Omega_q^{\nu,\kappa} : \mathbb{R}_+ \rightarrow \mathbb{R}$  given by

$$\Omega_q^{\nu,\kappa}(x) := \int_0^\infty \frac{\|\Phi_U^{\nu,\kappa}(\lambda x)\|_{\mathbb{C}^2}^2}{(1+\lambda)^q} d\lambda.$$

Employing (5.17) we conclude the existence of  $H_q^{\nu,\kappa} \in \mathbb{R}_+$  such that

$$\Omega_q^{\nu,\kappa} \leq H_q^{\nu,\kappa} ((\cdot)^{-2\beta} + 1).$$

Substituting into (6.10), computing the inner integral and minimising over  $q \in (1, 1 + \gamma - 2\beta)$  we obtain (1.7).  $\square$

## 7 Proof of Theorem 5

We will need the following lemma to control higher channels in the angular momentum decomposition.

**Lemma 21.** *Let  $\nu \in \mathbb{R}$ . For all*

$$\kappa \in \mathbb{Z} + 1/2 \quad \text{with} \quad |\kappa| \geq \kappa_\nu := \min\{\kappa \in \mathbb{N} + 1/2 : \kappa^2 > \nu^2 + 1/4\} \quad (7.1)$$

*the estimate*

$$|D_{\pi/2}^{\nu, \kappa}| \geq C^\nu |D_{\pi/2}^{0, \kappa}| \quad (7.2)$$

*holds with*

$$C^\nu := 1 - \frac{\nu^2(\kappa_\nu^2 + 1/4)}{(\kappa_\nu^2 - 1/4)^2} \left( \sqrt{1 + \frac{(4\kappa_\nu^2 - \nu^2)(\kappa_\nu^2 - 1/4)^2}{(\kappa_\nu^2 + 1/4)^2 \nu^2}} - 1 \right) > 0.$$

*Proof.* The case of  $\nu = 0$  is trivial. Otherwise we proceed as in the proof of Lemma 28 in [11]. The only difference is that we now have relaxed the assumptions on  $\kappa$  and  $\nu$ .

As explained in the original proof, the statement of the lemma holds true provided for  $b := 1 - C^\nu$  we have  $b < 1$  and the inequality

$$a_{\kappa, -}^\nu(b, s) := \nu^2 + b/4 + \kappa^2 b + s^2 b - (4\kappa^2 \nu^2 + 4\nu^2 s^2 + \kappa^2 b^2)^{1/2} \geq 0 \quad (7.3)$$

holds for all  $\kappa$  satisfying (7.1) and  $s \in \mathbb{R}$ . Since (7.3) does not depend on the signs of  $\kappa$  and  $s$ , without loss of generality we can assume  $\kappa, s \in \overline{\mathbb{R}_+}$ . Extending (7.3) to  $\kappa \in \mathbb{R}$ , we get for  $\kappa > 0$  satisfying (7.1)

$$\begin{aligned} a_{\kappa+1, -}^\nu(b, s) - a_{\kappa, -}^\nu(b, s) &= \int_\kappa^{\kappa+1} \frac{\partial a_{\varkappa, -}^\nu}{\partial \varkappa}(b, s) d\varkappa \\ &= (2\kappa + 1)b - \int_\kappa^{\kappa+1} \frac{(4\nu^2 + b^2)\varkappa}{\sqrt{(4\nu^2 + b^2)\varkappa^2 + 4\nu^2 s^2}} d\varkappa \geq (2\kappa_\nu + 1)b - \sqrt{4\nu^2 + b^2}, \end{aligned}$$

so for all other parameters being fixed,  $a_{\kappa, -}^\nu(b, s)$  is an increasing function of  $\kappa \geq \kappa_\nu$  provided

$$(2\kappa_\nu + 1)b - \sqrt{4\nu^2 + b^2} \geq 0,$$

i.e.

$$b \geq |\nu| / \sqrt{\kappa_\nu^2 + \kappa_\nu}$$

holds. For  $s > 0$  we have

$$\frac{\partial a_{\kappa_\nu, -}^\nu(b, s)}{\partial s} = 2s \left( b - \frac{2\nu^2}{\sqrt{4\kappa_\nu^2 \nu^2 + 4\nu^2 s^2 + \kappa_\nu^2 b^2}} \right) \geq 2s \left( b - \frac{2\nu^2}{\kappa_\nu \sqrt{4\nu^2 + b^2}} \right).$$

Thus, provided

$$b \geq \sqrt{2}|\nu|(\sqrt{1 + \kappa_\nu^{-2}} - 1)^{1/2} \quad \left( > |\nu| / \sqrt{\kappa_\nu^2 + \kappa_\nu} \right) \quad (7.4)$$

holds, for any  $s \in \mathbb{R}_+$  we have  $(\partial a_{\kappa_\nu, -}^\nu / \partial s)(b, s) \geq 0$  and hence for  $\kappa \geq \kappa_\nu$

$$a_{\kappa, -}^\nu(b, s) \geq a_{\kappa_\nu, -}^\nu(b, 0) = \nu^2 + b/4 + \kappa_\nu^2 b - \kappa_\nu(4\nu^2 + b^2)^{1/2} \quad (7.5)$$

holds. The right hand side of (7.5) is positive provided  $b > 0$  satisfies

$$f_\nu(b) := (\kappa_\nu^2 - 1/4)^2 b^2 + 2(\kappa_\nu^2 + 1/4)\nu^2 b + \nu^4 - 4\nu^2 \kappa_\nu^2 \geq 0. \quad (7.6)$$

Here  $f_\nu$  is a quadratic function with  $f_\nu(0) < 0$  and  $f_\nu(1) = (\kappa_\nu^2 - \nu^2 - 1/4)^2 > 0$ . Thus  $f_\nu$  has a unique zero in  $(0, 1)$  which coincides with  $1 - C^\nu$ . Hence for  $b > 0$  (7.6) is equivalent to  $b \geq 1 - C^\nu$ . Note that this condition is more restrictive than (7.4), since

$$\sqrt{2}|\nu|(\sqrt{1 + \kappa_\nu^{-2}} - 1)^{1/2} \leq |\nu|/\kappa_\nu$$

and by (7.1)

$$f_\nu(|\nu|/\kappa_\nu) \leq \nu^2(\kappa_\nu^{-2}/16 + \nu^2 - \kappa_\nu^2) \leq \nu^2(1/4 + \nu^2 - \kappa_\nu^2) < 0$$

holds.  $\square$

*Proof of Theorem 5, part 1.* For any  $\psi \in P_{[0, \infty)}(D_{\theta(\kappa_0)}^{\nu, \kappa_0})\mathfrak{D}(D_{\theta(\kappa_0)}^{\nu, \kappa_0})$  we have

$$\Psi := \mathcal{A}^* \bigoplus_{\kappa \in \mathbb{Z} + 1/2} \delta_{\kappa, \kappa_0} \psi \in P_{[0, \infty)}(D_\theta^\nu)\mathfrak{D}(D_\theta^\nu)$$

(here  $\delta_{\kappa, \kappa_0}$  is a Kronecker delta) and by (2.9) and (2.11)

$$\langle \Psi, D_\theta^\nu(Q)\Psi \rangle_{P_{[0, \infty)}(D_\theta^\nu)\mathcal{L}^2(\mathbb{R}^2, \mathbb{C}^2)} = \langle \psi, D_{\theta(\kappa_0)}^{\nu, \kappa_0}(V)\psi \rangle_{P_{[0, \infty)}(D_{\theta(\kappa_0)}^{\nu, \kappa_0})\mathcal{L}^2(\mathbb{R}_+, \mathbb{C}^2)} \quad (7.7)$$

holds with  $V$  as defined in (2.13). But since all the assumptions of Theorem 3, part I are satisfied for  $D_{\theta(\kappa_0)}^{\nu, \kappa_0}(V)$ , we can choose  $\psi$  so that (7.7) is negative. Hence by the minimax principle  $D_\theta^\nu(Q)$  has non-empty negative spectrum.  $\square$

*Proof of Theorem 5, part 2.* Given  $Q$  satisfying (2.14) and  $\alpha \in \mathbb{R}_+$ , we have

$$D_\theta^\nu(\alpha Q) \geq D_\theta^\nu(\alpha R(|\cdot|)\mathbb{I}) = \mathcal{A}^* \left( \bigoplus_{\kappa \in \mathbb{Z} + 1/2} D_{\theta(\kappa)}^{\nu, \kappa}(\alpha R\mathbb{I}) \right) \mathcal{A}. \quad (7.8)$$

It is thus enough to show that every term on the right hand side of (7.8) has no negative spectrum for  $\alpha \in [0, \alpha_c]$  with  $\alpha_c > 0$  independent of  $\kappa$ .

For all  $\kappa \in \mathbb{Z} + 1/2$  with  $\kappa^2 \leq \nu^2 + 1/4$  assumption (a) or (b) and Theorem 3, part II or III, respectively, imply the existence of  $\alpha_\kappa > 0$  such that

$$D_{\theta(\kappa)}^{\nu, \kappa}(\alpha R\mathbb{I}) \geq 0 \quad (7.9)$$

holds for  $\alpha \in [0, \alpha_\kappa]$ .

Let us now consider  $\kappa$  satisfying (7.1).

Assumption (c) of the theorem means that there exist  $R_1 \in \mathcal{L}^\infty(\mathbb{R}_+, r dr)$  and  $R_2 \in \mathcal{L}^2(\mathbb{R}_+, r dr)$  such that  $R = R_1 + R_2$ . We have

$$D_{\theta(\kappa)}^{\nu, \kappa}(\alpha R\mathbb{I}) = \frac{1}{2} (D_{\theta(\kappa)}^{\nu, \kappa}(2\alpha R_1\mathbb{I}) + D_{\theta(\kappa)}^{\nu, \kappa}(2\alpha R_2\mathbb{I})). \quad (7.10)$$

Theorem 2.5 in [8] implies

$$\mathcal{A}^* \left( \bigoplus_{\kappa \in \mathbb{Z} + 1/2} |D_{\pi/2}^{\kappa, 0}| \right) \mathcal{A} = (-\Delta)^{1/2} \mathbb{I} \geq K |\cdot|^{-1} \mathbb{I} = \mathcal{A}^* \left( \bigoplus_{\kappa \in \mathbb{Z} + 1/2} K/(\cdot) \mathbb{I} \right) \mathcal{A}$$

with

$$K := 2 \frac{\Gamma^2(3/4)}{\Gamma^2(1/4)},$$

from which we conclude

$$|D_{\pi/2}^{\kappa, 0}| \geq K/(\cdot), \quad \text{for all } \kappa \in \mathbb{Z} + 1/2. \quad (7.11)$$

For all  $\kappa$  satisfying (7.1) by Theorem 1 we have  $\theta(\kappa) = \pi/2$ . Combining this with (7.2) and (7.11) we obtain

$$|D_{\theta(\kappa)}^{\nu, \kappa}| \geq C^\nu K/(\cdot)$$

for all  $\kappa$  satisfying (7.1). Since  $R_1 \in \mathbf{L}^\infty(\mathbb{R}_+, r dr)$ , there exists  $\alpha_0 > 0$  such that for all  $\kappa$  satisfying (7.1)

$$D_{\theta(\kappa)}^{\nu, \kappa}(2\alpha R_1 \mathbb{I}) \geq 0 \quad (7.12)$$

holds for  $\alpha \in [0, \alpha_0)$ .

For  $R_2$  we use (7.2) to obtain

$$\begin{aligned} \text{rank } P_{(-\infty, 0)}(D_{\theta(\kappa)}^{\nu, \kappa}(2\alpha R_2 \mathbb{I})) &\leq \text{rank } P_{(-\infty, 0)}(|D_{\pi/2}^{\kappa, 0}| - 2(C^\nu)^{-1} \alpha R_2 \mathbb{I}) \\ &\leq \sum_{\kappa \in \mathbb{Z} + 1/2} \text{rank } P_{(-\infty, 0)}(|D_{\pi/2}^{0, \kappa}| - 2(C^\nu)^{-1} \alpha R_2 \mathbb{I}) \\ &= \text{rank } P_{(-\infty, 0)}\left((-\Delta)^{1/2} \mathbb{I} - 2(C^\nu)^{-1} \alpha R_2(|\cdot|) \mathbb{I}\right) \\ &= 2 \text{rank } P_{(-\infty, 0)}\left((-\Delta)^{1/2} - 2(C^\nu)^{-1} \alpha R_2(|\cdot|)\right). \end{aligned} \quad (7.13)$$

Now we invoke the Cwikel-Lieb-Rozenblum inequality for  $(-\Delta)^{1/2}$  in  $\mathbb{R}^2$ : By Remark 2.5 in [2] (or Example 3.3 in [5]) there exists  $C_{\text{CLR}} > 0$  such that the right hand side of (7.13) does not exceed

$$2C_{\text{CLR}} \|R_2(|\cdot|)\|_{\mathbf{L}^2(\mathbb{R}^2)}^2 = 16\pi(C^\nu)^{-2} \alpha^2 C_{\text{CLR}} \int_0^\infty R_2^2(r) r \, dr =: (\alpha/\alpha_1)^2.$$

Thus for  $\alpha \in [0, \alpha_1)$  the operator  $D_{\theta(\kappa)}^{\nu, \kappa}(2\alpha R_2 \mathbb{I})$  has no negative spectrum. Note that  $\alpha_1$  does not depend on  $\kappa$ .

Letting

$$\alpha_c := \min(\{\alpha_0, \alpha_1\} \cup \{\alpha_\kappa : \kappa \in \mathbb{Z} + 1/2, \kappa^2 \leq \nu^2 + 1/4\}) > 0$$

and combining the last result with (7.12) and (7.10) we observe that (7.9) holds for all  $\kappa \in \mathbb{Z} + 1/2$  provided  $\alpha \in [0, \alpha_c)$ .  $\square$

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